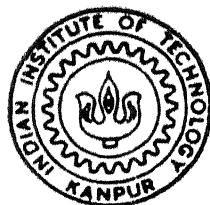


# OPTIMAL MULTISTEP METHODS FOR SECOND ORDER INITIAL AND BOUNDARY VALUE PROBLEMS

*by*

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**DEPARTMENT OF MATHEMATICS**

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**February 1994**

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**OPTIMAL MULTISTEP METHODS  
FOR SECOND ORDER INITIAL AND  
BOUNDARY VALUE PROBLEMS**

**A Thesis Submitted  
in Partial Fulfilment of the Requirements  
for the Degree of  
DOCTOR OF PHILOSOPHY**

**By  
RATNA BHATTACHARYA**

**to the  
DEPARTMENT OF MATHEMATICS  
INDIAN INSTITUTE OF TECHNOLOGY  
KANPUR**

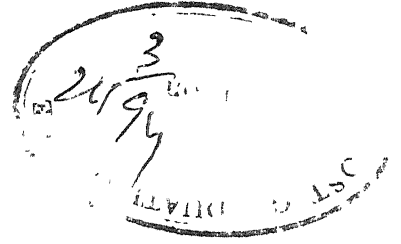
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## Certificate

It is certified that the work contained in this thesis entitled "**Optimal Multistep Methods for Second Order Initial and Boundary Value Problems**", by Ms. Ratna Bhattacharya, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree or diploma.

( R. K. S. Rathore )

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February, 1994



*To*  
*My Parents*

## ACKNOWLEDGEMENTS

I wish to express my deep and sincere thanks to Professor R. K. S. Rathore for his guidance and encouragement throughout this thesis work. Thanks are due to Mrs.(Dr.) Rathore for her tolerance, during my long discussions with guide at their home.

I am thankful to my family members specially to my husband.

I would like to express my thanks to all faculty members and office staff in our department.

My thanks goes to my friends and colleagues, specially to Rangan, Joydip, Pratibha, Nisha, Snigdha, Subir, Amit.

Finally thanks goes to Mr. Metia for his careful printing.

February, 1994

Ratna Bhattacharya

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## INTRODUCTION

An initial value problem of the form

$$(1) \quad y'' = f(x, y), \quad y(a) = A_0, \quad y'(a) = B_0,$$

or a boundary value problem of the form

$$(2) \quad y'' + f(x, y) = 0, \quad y(a) = A, \quad y(b) = B,$$

in which  $f$  is a function of  $x$  and  $y$  and not of  $y'$ , and is continuous and Lipchitzian with respect to the second argument, occur frequently in various fields, e.g., in mechanical problems without dissipation and in astronomical problems. In the words of Henrici [96], "Indeed astronomers have for over a century used methods for integrating this kind of differential equations which are of the multistep type and which work without first derivatives."

A classical Linear  $k$ -step method of general form is given by

$$(3) \quad \sum_{i=0}^k \alpha_i y_{n+k-i} + h^2 \sum_{i=0}^k \beta_i f_{n+k-i} = 0,$$

where  $y_j = y(x_j)$ ,  $f_j = f(x_j, y_j)$ ,  $x_j = a + jh$ , and  $h$ ,  $\alpha_j$ ,  $\beta_j$  are constants, which do not depend on  $n$ . In such a formula, the coefficients are chosen in such a way that the associated linear difference operator given by

$$(4) \quad L[y(x); h] \equiv \sum_{i=0}^k \alpha_i y(x+(n+k-i)h) + h^2 \sum_{i=0}^k \beta_i y''(x+(n+k-i)h),$$

is small for a solution of  $y'' = f(x, y)$ , that means, the method is made locally exact for polynomials of highest possible degree. The precision with respect to polynomial functions renders the

coefficients  $\alpha_1$ 's,  $\beta_1$ 's independent of  $x_n$ . This property of coefficients is retained even when the formulae are made interpolatory for a maximal set of exponential functions  $e^{\alpha x}$ 's. However, if the formulae are made interpolatory with respect to a general class of functions, the coefficients  $\alpha_1$ 's, and  $\beta_1$ 's would no more be independent of the point  $x_n$  under consideration.

To solve an initial value problems of the form (1), one can use linear multistep methods with constant coefficients. Indeed such methods are to be (i) Stormer's method (ii) Cowell's method, or some further special methods discussed by Collatz [53], Henrici [96], Jain [100], Lapidus and Seinfeld [125] and some others, provided with a special starting procedure. Many researchers have considered linear multistep methods for solving initial value problems and have discussed their convergence, consistency and stability criteria. Cash [31] has derived exponentially fitted multiderivative linear multistep methods of orders upto 5. He has considered the class of integration formulae, each containing a "built-in" local error estimate - a facility lacking in most other exponentially fitted formulae. Verwer [191], [192] has proposed a 3-rd order formula for the numerical integration of stiff systems of ordinary differential equations for first order initial value problems. This formula belongs to a special class of generalized linear multistep method formulae, of which the scalar coefficients are replaced by operator coefficients. Cooper [55] has considered a general linear multistep method for first order initial value problem and its order of convergence.

Theorems on linear multistep methods with constant stepsize

and constant formula were developed after the publication of Dahlquist's classical paper [58]. After 1969, linear multistep formulae have often been applied in practical codes as variable step size variable formula methods (VSVFM). Zlatev [195] has considered a general LMVSVFM of the form :

$$y_k = \sum_{i=1}^{s_k} \alpha_i(\bar{h}_k, s_k) y_{k-i} + \sum_{i=0}^{s_k} h_{k-i} \beta_i(\bar{h}_k, s_k) f(x_{k-i}, y_{k-i})$$

where  $h_k = x_k - x_{k-1}$ ,  $\bar{h}_k = (h_k, h_{k-1}, \dots, h_{k-s_k})$ ,  $s_k \leq k$ ,  $k = 1(1)N$ .

The coefficients  $\alpha_i$ 's and  $\beta_i$ 's depend on the last  $s_k+1$  and on the formula used at step  $k$ . He has defined a class of 3 ordinate LMVSVFM and has obtained some results concerning the zero-stability properties of these methods. Zlatev [199] has proved a formal definition of general LMVSVFM's and has shown some theorems concerning the consistency and the convergence of these methods. Marz Roswitha [132]-[134] has discussed various kinds of variable multistep methods and has shown several facts concerning stability, consistency and convergence of general two point boundary value problems of ordinary differential equations and has treated initial value problems as special cases.

Many authors have discussed stability criteria for linear multistep methods for ordinary differential equations. Dahlquist [59] in 1963, had defined A-stability for a linear multistep method with constant coefficients, applied to a first order differential equation and had proved the well known result that the order of an A-stable linear multistep method cannot exceed 2. Dahlquist [60] showed that a norm (Liapunov function) can be constructed for the stability and the error analysis of a linear multistep method for

the solution of a stiff non-linear system, provided that the system satisfies a monotonicity condition and the method possesses a property called G-stability. Dahlquist [62] has shown that G-stability and A-stability concepts are equivalent for multistep methods in their one leg formulation. Hairer and Turke [94] have shown to what extent this result also holds for Runge-Kutta methods. Veldhuizen [190] has proposed D-stability criterion for discretization methods for stiff, first order initial value problems. He has shown that D-stability is a necessary requirement for stability in the usual sense. Liniger [130] has given two easy-to-check conditions which, together, are sufficient and "almost necessary" for A-stability of linear multistep integration formulae. Most of the literature on stability of a numerical solution of ordinary differential equations is concerned with absolute stability, which tells whether the extraneous solutions introduced in the numerical solution grow or decay in magnitude. Because of difficulty in finding a high order absolutely stable numerical solution, researchers paid their attention to relative stability, which is a relation of the magnitude of the extraneous solutions to that of the true solution. Relative stability is mentioned by Hamming [95]. Ralston [153] has given a formal definition of relative stability of a numerical method for first order initial value problems and a simple computational scheme by which this definition can be tested. In order to solve initial value problems for systems of first order differential equations, where Jacobian of the right hand side of a linear multistep method has purely imaginary eigen values, the method is called stable



along the imaginary axis, according to Jeltsch [107] and Dekker [72]. Odeh and Liniger [145] have established nonlinear fixed  $-h$  stability similar to A-stability for implicit linear multistep formulae, when applied to nonlinear stable systems and have given sufficient criteria which guarantee the fixed  $-h$  stability of the global numerical error for various classes of non-linearities. Jeltsch [102], [105], [108], [109] and Bickart [24] have investigated some other stability criteria viz.  $A_0$ -stability,  $A(0)$ -stability,  $A[\alpha]$ -stability and stiff stability for solving first order initial value problem. Many other authors viz., Chen [51], Cryer [57], Friedli and Jeltsch [85], Gear et. al. [88], [88], Grigorieff [93], Koloza [116], [117], Li Wangyaa [129], Nørsett [143], Oliveira and Patricio [147], Wanner et. al. [193]-[195], Widlund [196] have discussed linear multistep methods with constant coefficients for solving initial value problems and their various stability criteria.

Many researchers have used finite difference methods for solving two point boundary value problems. Asfar and Hussein [5], Tewarson et. al. [179], [181], [182] have considered high order numerical methods by writing the differential equation as a system of first order differential equations with general two point boundary conditions, with the help of the methodology of finite differences, splines and fundamental matrix. Many authors including Boutayeb and Twizell [25], Cash [32], Chawla [39]-[42], Tewarson [180], Twizell [184], Twizell and Boutayeb [185] have constructed high order finite difference schemes for solving two point boundary value problems with natural and mixed boundary conditions. Baboo

[7] and some other authors have considered finite difference methods for solving two point boundary value problems with periodic boundary conditions.

Much work has been done on finite difference methods for solving singular two point boundary value problems. Chawla and Katti [44]-[47] have considered a three point finite difference method for the singular two point boundary value problem of the form  $y'' + (2/x)y' + f(x,y) = 0$ ,  $y'(0) = 0$ ,  $y(1) = a$  replacing  $y'$  and  $y''$  by difference approximations. Chawla, Mckee and Shaw [48] have constructed a finite difference method based on uniform mesh for the problem  $Lu = x^\alpha f(x,u)$ ,  $u'(0) = 0$ ,  $u(1) = A$ , with  $\alpha \geq 1$  and have shown  $O(h^2)$  convergence of the method. Chawla [49] presents a new fourth order finite difference method for the weakly singular two point boundary value problem:  $Lu = f(x,u)$ , on  $(0,1]$  subject to boundary condition  $u(0) = A$ ,  $u(1) = B$  for  $0 < \alpha < 1$ . The method reduces to the fourth order Numerov method for  $u'' = f(x,u)$ . Katti and Chopra [111] have studied the stability of modified Numerov method of Jain et. al. [101] for the class of singular two point boundary value problems of the form  $y'' + (2/x)y' + f(x,y) = 0$ ,  $y'(0) = 0$ ,  $y(1) = a$ . Balajan and Molokovic [9], Ciarlet [52], De Hoog and Weiss [69]-[71], Doedel and Reddien [76]-[78], Jain and Jain [101], and Pandey [149] and many others have worked on singular two point boundary value problems.

An ordinary differential equation can be solved by multistep methods based on quadrature formulae with precision for certain functions. Many researchers have shown that instead of using the usual quadrature formulae, optimal quadrature formulae with minimum

error norm corresponding to various classes of functions, are more effective. Sard [165], Davis [64], Barnhill [10]-[13], [16]-[18], Barnhill and Wixom [15], Barnhill and Nelson [19], Barrer [20] and Larkin [126] have considered minimization of quadrature error in various manners with respect to weights to obtain the optimal quadrature formulae. Pinkus [150], Rabinowitz and Ritcher [151] and [152], Ritcher [157]-[159] have studied properties of such optimal formulae. Chawla and Kaul [36]-[38] and Kaul [113] have considered such optimal quadrature rules subject to interpolatory conditions for polynomials of degree less than or equal to the number of nodes. Kaul [112] and Finney and Price Jr. [82] have studied optimal quadrature rules with interpolatory conditions for arbitrary functions.

Some authors have studied linear multistep methods with varying coefficients and their convergence and stability criteria. Lambert [121], in 1970, has considered a linear multistep method with mildly varying coefficients of the form

$$\sum_{i=0}^k (\alpha_i + h a_j(x_n)) y_{n+k-i} = h^2 \sum_{i=0}^k (\beta_i + h b_j(x_n)) f_{n+k-i},$$

where  $a_j(x_n)$  and  $b_j(x_n)$  are functions specified by the particular differential equations under consideration, for solving a first order differential equation instead of using a constant method. He has shown that it is possible to control the weak stability characteristics of the method with varying coefficients. The weak instability associated with stable linear  $k$ -step usual method with order  $k+2$  can be removed by using the linear multistep methods with varying coefficients, under some stabilizing conditions. We are

thus motivated to study a class of linear multistep methods with varying coefficients. Lambert and Sigurdsson [124] have shown that the application of a generalization of this stabilizing condition to a somewhat linear multistep method is applicable to systems of differential equations. Sanz-Serna [164] has proved some further results for the general class of Lambert-Sigurdsson method. Aguado and Correias [1], Aguado [2], Andreassen [3], Correias [54], Cronzeix and Lisbona [56], Marz [132]-[134], Varil'kov and Zagonov [189], Zlatev [197], [198], Sigurdsson [172] and some others have considered linear multistep methods with varying coefficients for solving differential equations of first order or systems of differential equations of first order.

Brij Bhusan [27] in 1986, has studied optimal multistep methods for solving first order differential equations. His thesis consists of a study of such methods which minimize the quadrature error or the local truncation error functional over a Hilbert space possessing a reproducing kernel function, in the numerical integration of a first order initial value problem. Such kind of multistep methods turn out to be multistep methods with varying coefficients.

No similar approach to get optimal fomulae in numerical solution of second order initial and two point boundary value problems, however, seems to have been taken up in the literature even though a need for this has been felt for quite some time (Gear [90]).

This thesis is an approach in this direction and consists of a study of the optimal multistep methods which minimize the norm of

local truncation error functional in the numerical integration of a second order initial value problem or a boundary value problem.

The body of this thesis consists of six chapters. The layout of each chapter is given below.

## CHAPTER 1

This chapter is devoted to the derivation of various optimal multistep formulae in a general Hilbert space possessing a reproducing kernel function and the characterization of these formulae. The formulae with interpolatory constraints have been divided into two categories, one with interpolatory functions as polynomials of certain degree and the other with interpolatory functions being arbitrary.

It is shown that at each step, the coefficients of the optimal formulae give rise to a deterministic system of linear equations and that each formulae is characterized by it being locally interpolatory for a certain set of functions.

## CHAPTER 2

This chapter deals with the convergence and the theoretical error analysis of optimal multistep formulae adopted for second order initial value problems. We first determine a relation between the  $\beta$ -coefficients of the  $\beta$ -optimal and the corresponding usual methods. The convergence of  $\beta$ -optimal method has been established next and an error bound for discretization error and an *a priori* bound for round off error of this method has been devised, the approach being delineated by Henrici. Then we study general linear multistep methods with varying coefficients for second order

initial value problems, the basic motivation coming from the work of Lambert on multistep methods with varying coefficients for first order initial value problems and the fact that our optimal methods are also of this category. We have established the convergence of such methods and derived the bounds for discretization error and propagation of round off errors. We have also given a stability analysis of linear multistep methods with varying coefficients.

### CHAPTER 3

This chapter deals with some theoretical aspects of the  $\beta$ -optimal difference formulae corresponding to Cowell's usual formula with function evaluation at three points and Stormer's usual formula with function evaluation at one point, for solving two point boundary value problems of the form  $y'' + f(x,y) = 0$ ,  $y(a) = A$ ,  $y(b) = B$ , with  $-\infty < \partial f / \partial y < \pi^2 / (b-a)^2$ . We establish the convergence of such an optimal difference method using the properties of monotone matrices and the convergence of Newton's method for systems of nonlinear optimal difference equations. Next, we establish stability theory for  $\beta$ -optimal and corresponding usual methods.

### CHAPTERS 4,5 & 6

These chapters consist of numerical implementation of various optimal multistep formulae, derived in Chapter 1, adopted for second order initial and boundary value problems in three typical Hilbert spaces, viz.  $H^2(C_r)$ ,  $L^2(\hat{C}_r)$  and  $H_{a,b}^{(m)}$  respectively. In  $H^2(C_r)$  and  $L^2(\hat{C}_r)$  spaces known reproducing Kernel functions have been used. The Kernel function for  $H_{a,b}^{(m)}$  space has been derived.

Each of these chapters contains (i) the appropriate system of

linear equations, particularly simplified for the respective spaces, to determine the optimal coefficients; (ii) the locally interpolatory characterization of various optimal methods; and (iii) tabulated numerical results for various optimal methods for both initial and boundary value problems.

In case of optimal methods implemented for initial value problems, we use Cowell's three points and Stormers five points methods, and in case of optimal methods implemented for boundary value problems we use Stormers one point and Cowell's three points methods.

In chapters 4 and 5 we have also studied the limiting behaviour of the optimal  $\beta$ -coefficients as  $r \rightarrow \infty$ . It has been shown that, the optimal  $\beta$ -coefficients have the limiting values equaling the  $\beta$ -coefficients of the corresponding usual method with maximal polynomial precision, with the same  $\alpha$ -coefficients and the interpolatory constraints.

# CHAPTER 1

## OPTIMAL MULTISTEP METHODS FOR SPECIAL SECOND ORDER DIFFERENTIAL EQUATIONS

### 1.1 Introduction

Let us consider an initial value problem (IVP) with a special second order differential equation of the form

$$(1) \quad y'' = f(x, y), \quad y(a) = A_0, \quad y'(a) = B_0,$$

where the function  $f(x, y)$  satisfies the following conditions:

(i)  $f(x, y)$  is defined and continuous in the strip  $a \leq x \leq b$ ,  $-\infty < y < \infty$ , where  $a$  and  $b$  are finite.

(ii) There exists a constant  $L$  such that for any  $x \in [a, b]$  and any two numbers  $y$  and  $y^*$

$$|f(x, y) - f(x, y^*)| \leq L |y - y^*|$$

For the numerical solution of the IVP (1) we introduce the points  $x_n = a + nh$ ,  $0 \leq n \leq N$ , where  $h = (b-a)N^{-1}$  and  $N$  is an appropriate integer. A linear multistep method with  $k$  steps,  $k \geq 2$  of the form

$$(2) \quad \sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0,$$

where  $f_m = f(x_m, y_m)$  can be designed for the determination of the numbers  $y_n$  numerically which, it is hoped, approximate the values  $y(x_n)$  of the true solution at the points  $x_n$ . We shall assume that  $\alpha_0 \neq 0$ ,  $|\alpha_k| + |\beta_k| > 0$ . Here  $\delta_{t_0}$  is the Kronecker delta so that



the method (2) is an explicit method if  $t = 0$ , and is an implicit method if  $t \neq 0$ , say  $t = 1$ .

With the difference equation (2) for the differential equation (1), we associate the difference operator given by

$$(3) \quad L[y(x);h] = \sum_{i=0}^k \alpha_i y(x+(k-i)h) - h^2 \sum_{i=\delta_{t0}}^k \beta_i y''(x+(k-i)h).$$

The coefficients  $\alpha_i$ 's and  $\beta_i$ 's are chosen in such a way that  $L[y(x);h]$  is small for a solution of the given differential equation (1).

The present chapter is devoted to the derivation of various optimal formulae and their characterization. We shall develop optimal multistep methods of various forms which can be adopted for solving an initial value problem in such a way that the norm of the local truncation error functional of the corresponding method is pointwise minimum at each step over a Hilbert space  $H$ , possessing a reproducing kernel function.

Let  $D$  denote a point set contained in the space of a real or complex variable, and let  $H$  be a Hilbert space of functions  $f(x)$ , the domain of each  $f \in H$  being  $D$ . If there exists a function  $K(x, \bar{y})$  of two variables  $x, y \in D$ , which satisfies the inner product relation

$$(4) \quad f(y) = (f(x), K(x, \bar{y})), \quad f \in H, y \in D,$$

and also, for any fixed  $y \in D$ ,  $K(x, \bar{y}) \in H$ , regarded as a function of  $x$ , then  $K$  is said to be a reproducing kernel function for  $H$  (Aronszajn [4], Larkin [126]). Here the bar denotes complex conjugate.

A necessary and sufficient condition that  $H$  possesses a

reproducing kernel function is that, for every fixed  $x \in D$ , the linear functional  $Lf = f(x)$  is bounded. That is, there exists a finite constant  $C_x$ , depending upon  $x$ , such that

$$(5) \quad |f(x)| \leq C_x \|f\|, \quad \text{for all } f \in H.$$

This is a well known result due to Aronszajn [4].

Furthermore, the reproducing kernel function for a Hilbert space, if it exists, is unique and satisfies the relation

$$(6) \quad K(y, \bar{x}) = \overline{K(x, \bar{y})}, \quad \text{for all } x, y \in D.$$

In this chapter, we shall assume that the  $k$ -th derivative evaluation functional  $Dk_x : f \rightarrow f^k(x)$ , for  $k = 2$ , is a bounded linear functional in  $H$  for every  $x \in D$ . Further, we shall assume that  $H$  contains, as a subspace, the set of all polynomial functions.

In section 1.2, we obtain and characterize optimal multistep methods which are obtained by minimizing the local truncation error functional over a Hilbert space possessing a reproducing kernel function.

In section 1.3, we make a general multistep method exact for polynomials upto certain degree and find the corresponding optimal multistep method by minimizing the norm of local truncation error functional over a Hilbert space possessing a reproducing kernel function.

In section 1.4, interpolatory conditions are developed for a general set of preassigned functions subject to which the norm of local truncation error functional is minimized over a Hilbert space. The optimal methods obtained are shown to get characterized by interpolatory conditions for a certain set of functions.

In recent work, many researchers have considered boundary value problems (BVP) with a special second order differential equation of the form

$$(7) \quad y'' + f(x, y) = 0, \quad y(a) = A, \quad y(b) = B,$$

where the function  $f(x, y)$ , in addition to satisfying the conditions (i) and (ii), have continuous partial derivatives with respect to  $y$ . Assume that

$$(8) \quad u_* = \inf_{a \leq x \leq b} \frac{\partial f}{\partial y} \quad \text{and} \quad u^* = \sup_{a \leq x \leq b} \frac{\partial f}{\partial y},$$

then we shall study two categories of boundary value problems with either,  $-\infty < u^* \leq 0$ , or,  $0 < u^* < \pi^2/(b-a)^2$ .

For solving a BVP of the form (7) one can use a multistep method of the form (2) with the associated difference operator of the form (3) in which  $f$  is to be replaced by  $-f$ . The optimal methods to be derived in this chapter remain applicable in solving such BVP's with the term  $f$  replaced by  $-f$ . However, for a BVP (7) it is not necessary to assume that  $\alpha_0 \neq 0$  in (2).

In a multistep method (2), one or more  $\alpha$  or  $\beta$  coefficients could be zero. Then we would like those  $\alpha$  or  $\beta$  coefficients not to be present in the corresponding optimal methods.

## 1.2 Optimal Multistep Methods for 2-nd Order Differential Equations

Definition 1: Ignoring the round-off errors, let the numerical solution of the differential equation (1) satisfy

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta}^k \beta_i f_{n+k-i} = 0.$$

The true solution of the differential equation (1) satisfies

$$\sum_{i=0}^k \alpha_i Y(x_{n+k-i}) - h^2 \sum_{i=\delta_{t0}}^k \beta_i Y''(x_{n+k-i}) + T_n = 0,$$

where  $T_n$  is called the local truncation error at the  $n$ -th step (Jain [100]).

We shall assume that the 2-nd derivative evaluation functional is a bounded linear functional in  $H$ , with representer given by

$$(9) \quad D2(t, \bar{x}_{n+k-1}) = \overline{\frac{\partial^2}{\partial z^2} K(z, \bar{t})} \Big|_{z=x_{n+k-1}} = \frac{\partial^2}{\partial z^2} K(t, \bar{z}) \Big|_{z=x_{n+k-1}}.$$

Now we shall establish the following result.

Lemma 1: Let  $S = \{ K(t, \bar{x}_{n+k-1}), D2(t, \bar{x}_{n+k-1}) : i = 0 (1) k \}$ , for distinct points  $x_n, x_{n+1}, \dots, x_{n+k}$ . Then  $S$  is a linearly independent set.

Proof: We know if  $x_{n+k-1}, i = 0 (1) k$  are  $k+1$  distinct points, then the basic generalized Hermite interpolation polynomials  $P_j(x), Q_j(x), R_j(x), j = 0 (1) k$ , which are polynomials of degree  $3k+2$ , satisfy

$$\begin{aligned} P_j(x_{n+1-i}) &= \delta_{ij}, & Q_j(x_{n+1-i}) &= 0, & R_j(x_{n+1-i}) &= 0, \\ P'_j(x_{n+1-i}) &= 0, & Q'_j(x_{n+1-i}) &= \delta_{ij}, & R'_j(x_{n+1-i}) &= 0, \\ P''_j(x_{n+1-i}) &= 0, & Q''_j(x_{n+1-i}) &= 0, & R''_j(x_{n+1-i}) &= \delta_{ij}, \end{aligned}$$

for  $i = 0 (1) k, j = 0 (1) k$ .

Now let

$$\sum_{i=0}^k c_i K(t, \bar{x}_{n+k-1}) + \sum_{i=0}^k d_i D2(t, \bar{x}_{n+k-1}) = 0.$$

Then taking inner product with  $P_j(t)$  and  $R_j(t)$ ,  $j = 0(1)k$ , we get

$$\sum_{i=0}^k c_i P_j(x_{n+k-1}) + \sum_{i=0}^k d_i P_j''(x_{n+k-1}) = 0, \quad j = 0(1)k,$$

which implies  $c_j = 0$ ,  $j = 0(1)k$ , and

$$\sum_{i=0}^k c_i R_j(x_{n+k-1}) + \sum_{i=0}^k d_i R_j''(x_{n+k-1}) = 0, \quad j = 0(1)k,$$

which implies  $d_j = 0$ ,  $j = 0(1)k$ . Hence the proof.

Now we shall find various optimal multistep methods.

Firstly, we shall find  $\beta$ -optimal method where optimization is done with respect to the  $\beta$ -coefficients while the  $\alpha$ -coefficients are prefixed as in the usual method (2).

Let

$$(10) \quad \sum_{i=0}^k \alpha_i y_{n+k-1} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} f_{n+k-1} = 0$$

be the  $\beta$ -optimal method corresponding to the usual method (2), where  $\hat{\beta}_{i,n}$  are the optimal coefficients corresponding to the usual coefficients  $\beta_i$ ,  $i = \delta_{t_0}(1)k$ , at a point  $x_n$ ,  $0 \leq n \leq N-k$ .

We consider the optimality criterion as the minimization of the norm of the local truncation error functional.

Now the local truncation error functional  $\hat{T}_n$  at a point  $x_n$  of the optimal method (10) applied to a function  $y(x)$ , a true solution of the differential equation (1), is given by

$$\hat{T}_n y = \sum_{i=0}^k \alpha_i y(x_{n+k-1}) - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} y''(x_{n+k-1}).$$

Therefore  $\|\hat{T}_n\|$  at a point  $x_n$  in a Hilbert space (H) is given by

$$\|\hat{T}_n\|^2 = \left\| \sum_{i=0}^k \bar{\alpha}_i K(t, \bar{x}_{n+k-1}) - h^2 \sum_{i=\delta_{t0}}^k \bar{\beta}_{i,n} D2(t, \bar{x}_{n+k-1}) \right\|^2,$$

where  $D2(t, \bar{x}_{n+k-1})$  designates the representer for the 2-nd derivative evaluation functional at  $x_{n+k-1}$ ,  $i = 0(1)k$ ;  $0 \leq n \leq N-k$ .

Since  $S = \{K(t, \bar{x}_{n+k-1}), D2(t, \bar{x}_{n+k-1}) : i=0(1)k\}$ , is linearly independent, to determine the optimal coefficients  $\hat{\beta}_{i,n}$  we shall minimize  $\|\hat{T}_n\|^2$  with respect to  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t0}(1)k$ ,  $0 \leq n \leq N-k$ . According to Larkin [126], the first order change  $\delta(\|\hat{T}_n\|^2)$ , following a change  $\delta(\hat{\beta}_{i,n})$ , in  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t0}(1)k$  is given by

$$\begin{aligned} \delta(\|\hat{T}_n\|^2) &= \\ &-h^2 \left( \delta(\bar{\beta}_{i,n}) D2(t, \bar{x}_{n+k-1}), \sum_{i=0}^k \bar{\alpha}_i K(t, \bar{x}_{n+k-1}) - h^2 \sum_{i=\delta_{t0}}^k \bar{\beta}_{i,n} D2(t, \bar{x}_{n+k-1}) \right) \\ &-h^2 \left( \sum_{j=0}^k \bar{\alpha}_j K(t, \bar{x}_{n+k-j}) - h^2 \sum_{j=\delta_{t0}}^k \bar{\beta}_{j,n} D2(t, \bar{x}_{n+k-j}), \delta(\bar{\beta}_{i,n}) D2(t, \bar{x}_{n+k-1}) \right) \\ &= -h^2 \overline{\delta(\hat{\beta}_{i,n})} \left[ \sum_{i=0}^k \alpha_i \left( D2(t, \bar{x}_{n+k-1}), K(t, \bar{x}_{n+k-1}) \right) \right. \\ &\quad \left. - h^2 \sum_{i=\delta_{t0}}^k \hat{\beta}_{i,n} \left( D2(t, \bar{x}_{n+k-1}), D2(t, \bar{x}_{n+k-1}) \right) \right] \\ &\quad - h^2 \delta(\hat{\beta}_{i,n}) \left[ \sum_{j=0}^k \bar{\alpha}_j \left( K(t, \bar{x}_{n+k-j}), D2(t, \bar{x}_{n+k-1}) \right) \right. \\ &\quad \left. - h^2 \sum_{j=\delta_{t0}}^k \bar{\beta}_{j,n} \left( D2(t, \bar{x}_{n+k-j}), D2(t, \bar{x}_{n+k-1}) \right) \right]. \end{aligned}$$

Now,

$$\left( D2(t, \bar{x}_{n+k-1}), K(t, \bar{x}_{n+k-1}) \right) = \frac{\partial^2}{\partial z^2} \left( K(t, \bar{z}), K(t, \bar{x}_{n+k-1}) \right) \Big|_{\bar{z}=\bar{x}_{n+k-1}}$$

$$= \frac{\partial^2}{\partial z^2} K(x_{n+k-1}, \bar{z}) \Big|_{z=x_{n+k-1}} = D2(x_{n+k-1}, \bar{x}_{n+k-1}).$$

Again,

$$\begin{aligned} \left( D2(t, \bar{x}_{n+k-1}), D2(t, \bar{x}_{n+k-1}) \right) &= \frac{\partial^2}{\partial z^2} \left( D2(t, \bar{x}_{n+k-1}), K(t, \bar{z}) \right) \Big|_{z=x_{n+k-1}} \\ &= \frac{\partial^2}{\partial z^2} \left( \frac{\partial^2}{\partial x^2} K(t, \bar{x}) \Big|_{x=x_{n+k-1}}, K(t, \bar{z}) \right) \Big|_{z=x_{n+k-1}} \\ &= \frac{\partial^2}{\partial z^2} \left[ \frac{\partial^2}{\partial x^2} K(z, \bar{x}) \Big|_{x=x_{n+k-1}} \right]_{z=x_{n+k-1}} \\ &= \frac{\partial^2}{\partial z^2} D2(z, \bar{x}_{n+k-1}) \Big|_{z=x_{n+k-1}} = D2''(x_{n+k-1}, \bar{x}_{n+k-1}), \text{ say.} \end{aligned}$$

Thus we get,

$$\begin{aligned} \delta(\|\hat{T}_n\|^2) &= \\ &- h^2 \overline{\delta(\hat{\beta}_{1,n})} \left[ \sum_{i=0}^k \alpha_i D2(x_{n+k-1}, \bar{x}_{n+k-1}) - h^2 \sum_{i=\delta_{t0}}^k \hat{\beta}_{i,n} D2''(x_{n+k-1}, \bar{x}_{n+k-1}) \right] \\ &- h^2 \delta(\hat{\beta}_{1,n}) \left[ \sum_{j=0}^k \bar{\alpha}_j \overline{D2(x_{n+k-j}, \bar{x}_{n+k-1})} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{j,n} \overline{D2''(x_{n+k-j}, \bar{x}_{n+k-1})} \right]. \end{aligned}$$

Since, for minimizing  $\hat{\beta}_{1,n}$ ,  $\delta(\|\hat{T}_n\|^2)$  is to vanish for an arbitrary choice of  $\delta(\hat{\beta}_{1,n})$ ,  $1 = \delta_{t0}(1)k$ ; we have

$$(11) \quad \sum_{i=0}^k \alpha_i D2(x_{n+k-1}, \bar{x}_{n+k-1}) - h^2 \sum_{i=\delta_{t0}}^k \hat{\beta}_{i,n} D2''(x_{n+k-1}, \bar{x}_{n+k-1}) = 0.$$

This is the system of normal equations solving which we can obtain  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t0}(1)k$ , at a point  $x_n$ .

Comparing (10) and (11), we have the following theorem.

Theorem 1: The optimal method (10), where  $\alpha_i$ ,  $i = 0(1)k$ , are prefixed and the optimization is done with respect to  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k$ , is characterized by that it is locally interpolatory for the functions

$$\{D2(x, \bar{x}_{n+k-1}), \quad 1 = \delta_{t_0}(1)k\}.$$

Now we shall find  $\alpha$ -optimal multistep method, where optimization is done with respect to the  $\alpha$ -coefficients, while the  $\beta$ -coefficients are prefixed as in the usual method (2). Let

$$(12) \quad \sum_{i=0}^k \hat{\alpha}_{i,n} y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0$$

be the  $\alpha$ -optimal method corresponding to the usual method (2), where  $\hat{\alpha}_{i,n}$  are the optimal coefficients corresponding to the usual coefficients  $\alpha_i$ ,  $i = 0(1)k$  at a point  $x_n$ ,  $0 \leq n \leq N-k$ .

We consider the optimality criterion as the minimization of the norm of the local truncation error functional.

Now the local truncation error functional  $\hat{T}_n^\alpha$  at a point  $x_n$  of the optimal method (12) applied to a function  $y(x)$ , a true solution of the differential equation (1), is given by

$$\hat{T}_n^\alpha y = \sum_{i=0}^k \hat{\alpha}_{i,n} y(x_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^k \beta_i y''(x_{n+k-i}).$$

As in Theorem 1,  $\|\hat{T}_n^\alpha\|^2$  at a point  $x_n$  in a Hilbert space  $H$  possessing a reproducing kernel function is given by

$$\|\hat{T}_n^\alpha\|^2 = \left\| \sum_{i=0}^k \bar{\alpha}_i K(t, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^k \bar{\beta}_i D2(t, \bar{x}_{n+k-i}) \right\|^2,$$

where  $D2(t, \bar{x}_{n+k-1})$  designates the representer for the second



derivative evaluation functional at  $x_{n+k-1}$ ,  $i = 0(1)k$ ,  $0 \leq n \leq N-k$ , which is defined in (9).

Since,  $S = \{K(t, \bar{x}_{n+k-1}), D2(t, \bar{x}_{n+k-1}) : i=0(1)k\}$ , is linearly independent, to determine the optimal coefficients  $\hat{\alpha}_{i,n}$ , we shall minimize  $\|\hat{T}_n^\alpha\|^2$  with respect to  $\hat{\alpha}_{i,n}$ ;  $i = 0(1)k$ , Proceeding as in Theorem 1, the first order change  $\delta(\|\hat{T}_n^\alpha\|^2)$ , following a change  $\delta(\hat{\alpha}_{1,n})$ , in  $\hat{\alpha}_{1,n}$ ,  $l = 0(1)k$  is given by

$$\delta(\|\hat{T}_n^\alpha\|^2) =$$

$$\begin{aligned} & \left( \delta(\hat{\alpha}_{1,n}) K(t, \bar{x}_{n+k-1}), \sum_{i=0}^k \bar{\alpha}_{i,n} K(t, \bar{x}_{n+k-1}) - h^2 \sum_{i=\delta_{t0}}^k \bar{\beta}_i D2(t, \bar{x}_{n+k-1}) \right) \\ & + \left( \sum_{j=0}^k \bar{\alpha}_{j,n} K(t, \bar{x}_{n+k-j}) - h^2 \sum_{j=\delta_{t0}}^k \bar{\beta}_j D2(t, \bar{x}_{n+k-j}), \delta(\hat{\alpha}_{1,n}) K(t, \bar{x}_{n+k-1}) \right) \\ & = \overline{\delta(\hat{\alpha}_{1,n})} \left[ \sum_{i=0}^k \hat{\alpha}_{i,n} \left( K(t, \bar{x}_{n+k-1}), K(t, \bar{x}_{n+k-1}) \right) \right. \\ & \quad \left. - h^2 \sum_{i=\delta_{t0}}^k \bar{\beta}_i \left( K(t, \bar{x}_{n+k-1}), D2(t, \bar{x}_{n+k-1}) \right) \right] \\ & \quad + \delta(\hat{\alpha}_{1,n}) \left[ \sum_{j=0}^k \bar{\alpha}_{j,n} \left( K(t, \bar{x}_{n+k-j}), K(t, \bar{x}_{n+k-1}) \right) \right. \\ & \quad \left. - h^2 \sum_{j=\delta_{t0}}^k \bar{\beta}_j \left( D2(t, \bar{x}_{n+k-j}), K(t, \bar{x}_{n+k-1}) \right) \right]. \end{aligned}$$

Since,  $\left( D2(t, \bar{x}_{n+k-j}), K(t, \bar{x}_{n+k-1}) \right) = D2(x_{n+k-1}, \bar{x}_{n+k-j})$ ,

$$\delta(\|\hat{T}_n^\alpha\|^2)$$

$$= \overline{\delta(\hat{\alpha}_{1,n})} \left[ \sum_{i=0}^k \hat{\alpha}_{i,n} \overline{K(x_{n+k-1}, \bar{x}_{n+k-1})} - h^2 \sum_{i=\delta_{t0}}^k \bar{\beta}_i \overline{D2(x_{n+k-1}, \bar{x}_{n+k-1})} \right]$$

$$+ \delta(\hat{\alpha}_{1,n}) \left[ \sum_{j=0}^k \bar{\alpha}_{j,n} K(x_{n+k-1}, \bar{x}_{n+k-j}) - h^2 \sum_{j=\delta_{t_0}}^k \bar{\beta}_j D2(x_{n+k-1}, \bar{x}_{n+k-j}) \right].$$

Since, for minimizing  $\hat{\alpha}_{1,n}$ ,  $\delta(\|\hat{T}_n^\alpha\|^2)$  is to vanish for an arbitrary choice of  $\delta(\hat{\alpha}_{1,n})$ ,  $l = 0(1)k$ , we have

$$\sum_{j=0}^k \bar{\alpha}_{j,n} K(x_{n+k-1}, \bar{x}_{n+k-j}) - h^2 \sum_{j=\delta_{t_0}}^k \bar{\beta}_j D2(x_{n+k-1}, \bar{x}_{n+k-j}) = 0.$$

Taking complex conjugate on both sides, we get

$$\sum_{j=0}^k \hat{\alpha}_{j,n} K(x_{n+k-j}, \bar{x}_{n+k-1}) - h^2 \sum_{j=\delta_{t_0}}^k \overline{\beta_j D2(x_{n+k-1}, \bar{x}_{n+k-j})} = 0,$$

or,

$$(13) \quad \sum_{j=0}^k \hat{\alpha}_{j,n} K(x_{n+k-j}, \bar{x}_{n+k-1}) - h^2 \sum_{j=\delta_{t_0}}^k \beta_j \frac{\partial^2}{\partial x^2} K(x, \bar{x}_{n+k-1}) \Big|_{x=x_{n+k-j}} = 0,$$

$$l = 0(1)k.$$

This is a system of normal equations, solving which we can obtain  $\hat{\alpha}_{j,n}$ ,  $j = 0(1)k$ , at a point  $x_n$ .

Comparing (12) and (13), we get the following theorem.

Theorem 2: The optimal method (12) where  $\beta_i$ ,  $i = \delta_{t_0}(1)k$ , are prefixed and the optimization is done with respect to  $\alpha_i$ 's,  $i=k(1)N-k$ , is characterized by that it is locally interpolatory for the functions

$$\{K(x, \bar{x}_{n+k-1}), \quad l = 0(1)k\}.$$

Now we shall find the  $\beta$ -optimal methods under some restriction. Let

$$\sum_{i=0}^k \alpha_i y_{n+k-1} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} f_{n+k-1} = 0$$

be the  $\beta$ -optimal method corresponding to the usual method (2). We

shall find the optimal coefficients  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k$ , at a point  $x_n$ ,  $0 \leq n \leq N-k$ , subject to the condition that

$$\sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} = 1.$$

Note that,  $\sum_{i=\delta_{t_0}}^k \beta_{i,n} = 1$ , is one of the consistency conditions

for a usual method (2). For convenience, let us write

$$\hat{\beta}_{k,n} = 1 - \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}.$$

Then the above  $\beta$ - optimal method becomes

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \left( \hat{\beta}_{k,n} f_n + \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n} f_{n+k-i} \right) = 0,$$

$$\text{or,} \quad \sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \left( f_n - \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n} f_n + \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n} f_{n+k-i} \right) = 0,$$

or,

$$(14) \quad \sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n} (f_{n+k-i} - f_n) = h^2 f_n.$$

The representer for the local truncation error functional for this method in a Hilbert space possessing a reproducing kernel function is given by

$$\hat{T}_n = \sum_{i=0}^k \bar{\alpha}_i K(t, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^{k-1} \bar{\beta}_{i,n} \left( D_2(t, \bar{x}_{n+k-i}) - D_2(t, \bar{x}_n) \right) - h^2 D_2(t, \bar{x}_n).$$

To obtain the optimal coefficients  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k-1$ , the normal equations can be obtained by minimizing  $\|\hat{T}_n\|^2$  with respect to  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k-1$ . Thus we obtain the normal equations

$$(h_1, \hat{T}_n) = 0, \quad 1 = \delta_{t_0}(1)k-1,$$

with  $h_1(t) = D2(t, \bar{x}_{n+k-1}) - D2(t, \bar{x}_n)$ , which gives the following system of equations:

$$(15) \quad \sum_{i=0}^k \alpha_i \left( D2(x_{n+k-1}, \bar{x}_{n+k-1}) - D2(x_{n+k-1}, \bar{x}_n) \right) \\ - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n} \left( D2''(x_{n+k-1}, \bar{x}_{n+k-1}) - D2''(x_{n+k-1}, \bar{x}_n) \right. \\ \left. - D2''(x_n, \bar{x}_{n+k-1}) + D2''(x_n, \bar{x}_n) \right) \\ - h^2 \left( D2''(x_n, \bar{x}_{n+k-1}) - D2''(x_n, \bar{x}_n) \right) = 0, \quad 1 = \delta_{t_0}(1)k-1.$$

This system of equations can be solved uniquely for  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k-1$ .

Thus we get the following theorem.

Theorem 3: The optimal multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-1} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} f_{n+k-1} = 0,$$

subject to the condition

$$\sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} = 1,$$

where  $\alpha_i$ ,  $i = 0(1)k$ , are prefixed and the optimization is done with respect to  $\beta_i$ 's,  $i = \delta_{t_0}(1)k-1$ , is characterized by that it is locally interpolatory for the functions

$$(h_1(x): 1 = \delta_{t_0}(1)k-1), \quad \text{where } h_1(x) = D2(x, \bar{x}_{n+k-1}) - D2(x, \bar{x}_n).$$

Now we shall find the  $\alpha$  as well as  $\beta$  optimal method under some

restriction. Let

$$(16) \quad \sum_{i=0}^k \hat{\alpha}_{i,n} y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} f_{n+k-i} = 0$$

be the  $\alpha$  as well  $\beta$  optimal method corresponding to the usual method (2), subject to the condition

$$\sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n} = 1.$$

We shall find the optimal coefficients  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$ , and  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k-1$  at a point  $x_n$ ,  $0 \leq n \leq N-k$ . For convenience, let

$$\hat{\beta}_{k,n} = 1 - \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}.$$

Then as in (14), the optimal method (16) becomes

$$(17) \quad \sum_{i=0}^k \hat{\alpha}_{i,n} y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n} (f_{n+k-i} - f_n) = h^2 f_n.$$

The representer for the local truncation error functional for this method in a Hilbert space possessing an appropriate reproducing kernel function is given by

$$\hat{T}_n = \sum_{i=0}^k \bar{\alpha}_{i,n} K(t, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n} (D_2(t, \bar{x}_{n+k-i}) - D_2(t, \bar{x}_n)) - h^2 D_2(t, \bar{x}_n).$$

To obtain the optimal coefficients  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$ , as well as  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k-1$ , the normal equations can be obtained by minimizing  $\|\hat{T}_n\|^2$  with respect to  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$ , and  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k-1$ . Thus we obtain the following normal equations

$$(h_1, \hat{T}_n) = 0, \quad i = \delta_{t_0}(1)k-1, \quad \text{where } h_1(x) = D_2(x, \bar{x}_{n+k-1}) - D_2(x, \bar{x}_n), \text{ and}$$

$$(g_1, \hat{T}_n) = 0, \quad l = 0(1)k, \quad \text{where } g_l(x) = K(x, \bar{x}_{n+k-l}).$$

Thus we get the following system of linear equations

$$(18) \quad \sum_{i=0}^k \hat{\alpha}_{i,n} K(x_{n+k-i}, \bar{x}_{n+k-l})$$

$$-h^2 \sum_{l=\delta_{t_0}}^{k-1} \hat{\beta}_{l,n} \left( \overline{D2(x_{n+k-l}, \bar{x}_{n+k-l}) - D2(x_{n+k-l}, \bar{x}_n)} \right) - h^2 \overline{D2(x_{n+k-l}, \bar{x}_n)} = 0,$$

$$l = 0(1)k, \text{ and}$$

$$(19) \quad \sum_{i=0}^k \hat{\alpha}_{i,n} \left( D2(x_{n+k-l}, \bar{x}_{n+k-l}) - D2(x_{n+k-l}, \bar{x}_n) \right) \\ - h^2 \sum_{l=\delta_{t_0}}^{k-1} \hat{\beta}_{l,n} \left( D2''(x_{n+k-l}, \bar{x}_{n+k-l}) - D2''(x_{n+k-l}, \bar{x}_n) \right. \\ \left. - D2''(x_n, \bar{x}_{n+k-l}) + D2''(x_n, \bar{x}_n) \right) \\ - h^2 \left( D2''(x_n, \bar{x}_{n+k-l}) - D2''(x_n, \bar{x}_n) \right) = 0, \quad l = \delta_{t_0}(1)k-1.$$

The system of equations (18) and (19) can be solved uniquely for  $\hat{\beta}_{l,n}$ ,  $l = \delta_{t_0}(1)k-1$  and  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$ .

Hence we get the following theorem.

Theorem 4: The optimal method

$$\sum_{i=0}^k \hat{\alpha}_{i,n} y_{n+k-l} - h^2 \sum_{l=\delta_{t_0}}^k \hat{\beta}_{l,n} f_{n+k-l} = 0,$$

where optimization is done with respect to  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$ ; as well as  $\hat{\beta}_{l,n}$ ,  $l = \delta_{t_0}(1)k$ , subject to the condition

$$\sum_{l=\delta_{t_0}}^k \hat{\beta}_{l,n} = 1,$$

is characterized by that it is locally interpolatory for the

functions,  $\{h_1(x): 1 = \delta_{t_0}(1)k-1\} \cup \{g_1(x): 1 = 0(1)k\}$ , where

$$h_1(x) = D_2(x, \bar{x}_{n+k-1}) - D_2(x, \bar{x}_n), \text{ and } g_1(x) = K(x, \bar{x}_{n+k-1}).$$

### 1.3 Optimal Multistep Methods Interpolatory for Polynomials

Definition 2: Let  $F(x)$  be a function which we wish to approximate using the class of functions  $\{g_n(x), n = 0, 1, 2, \dots\}$ . Let

$$F(x) \approx a_0 g_0(x) + a_1 g_1(x) + \dots + a_m g_m(x)$$

be an approximation of linear type to  $F(x)$ , where  $a_i, i = 0(1)m$ , are constants. We call an approximation exact or interpolatory, if the constants are chosen in such a way that on some fixed set of points  $\{x_i, i=1(1)p\}$ , the approximation and its first  $r_1$  derivatives, where  $r_1$  is a nonnegative integer, agree with  $F(x)$  except for round-off (Ralston and Rabinowitz [154]).

Firstly, we shall find the  $\beta$ - optimal multistep methods interpolatory for polynomials upto degree  $q$ .

The usual multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0$$

can be written, using divided difference notation, as

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \gamma_i y''[x_{n+k-i}, x_{n+k-i+1}, \dots, x_{n+k-\delta_{t_0}}] = 0.$$

If  $q \leq k + \delta_{t_1} + 1$ , we can choose the coefficients  $(\gamma_i, i = \delta_{t_0}(1) q + \delta_{t_0} - 2)$  in such a manner that the method is exact for polynomials upto degree  $q$ . Thus we have

$$\begin{aligned}
 (20) \quad & \sum_{l=0}^k \alpha_l y_{n+k-l} - h^2 \sum_{l=\delta_{t_0}}^{q+\delta_{t_0}-2} \gamma_l y''[x_{n+k-l}, x_{n+k-l+1}, \dots, x_{n+k-\delta_{t_0}}] \\
 & = h^2 \sum_{l=q+\delta_{t_0}-1}^k \gamma_l y''[x_{n+k-l}, x_{n+k-l+1}, \dots, x_{n+k-\delta_{t_0}}],
 \end{aligned}$$

in which optimization can be done with respect to the coefficients  $\{\gamma_l, l=q+\delta_{t_0}-1(1)k\}$ . So we shall minimize the norm of the local truncation error functional with respect to these coefficients.

Now, if  $D2(t, \bar{x})$  exists and is the representer for the 2-nd derivative evaluation functional in a Hilbert space  $H$ , then for any function  $f(x) \in H$ ,  $f''(x) = (f(t), D2(t, \bar{x}))$ . The divided difference  $y''[x_{n+k-l}, x_{n+k-l+1}, \dots, x_{n+k-\delta_{t_0}}]$  being a linear combination of  $y''(x_{n+k-l}), y''(x_{n+k-l+1}), \dots, y''(x_{n+k-\delta_{t_0}})$ , the representer, for  $y''[x_{n+k-l}, \dots, x_{n+k-\delta_{t_0}}]$  in  $H$ , would be  $D2[t; \bar{x}_{n+k-l}, \dots, \bar{x}_{n+k-\delta_{t_0}}]$ , which is obtained as a divided difference of  $D2(t, \bar{x})$  with respect to the 2-nd variable, which is a linear combination of  $D2(t, \bar{x}_{n+k-l}), D2(t, \bar{x}_{n+k-l+1}), \dots, D2(t, \bar{x}_{n+k-\delta_{t_0}})$ . Then we have for any function  $f(x) \in H$ ,

$$f''[x_{n+k-l}, \dots, x_{n+k-\delta_{t_0}}] = (f(t), D2[t; \bar{x}_{n+k-l}, \dots, \bar{x}_{n+k-\delta_{t_0}}])$$

The representer for the local truncation error functional  $\hat{T}_n^q$  of the optimal method (20), in a Hilbert space possessing a reproducing kernel function is given by

$$\hat{T}_n^q = \sum_{l=0}^k \bar{\alpha}_l K(t, \bar{x}_{n+k-l}) - h^2 \sum_{l=\delta_{t_0}}^{q+\delta_{t_0}-2} \bar{\gamma}_l D2[t, \bar{x}_{n+k-l}, \bar{x}_{n+k-l+1}, \dots, \bar{x}_{n+k-\delta_{t_0}}]$$



$$- h^2 \sum_{i=q+\delta_{t_0}-1}^k \bar{\gamma}_i D2[t, \bar{x}_{n+k-i}, \bar{x}_{n+k-i+1}, \dots, \bar{x}_{n+k-\delta_{t_0}}].$$

By Lemma 1,  $S = \{K(t, \bar{x}_{n+k-1}), D2[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k-\delta_{t_0}}] : i=0(1)k\}$  being linearly independent, its subset  $S_1 = \{K(t, \bar{x}_{n+k-1}), i=0(1)k; D2[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k-\delta_{t_0}}] : i=q+\delta_{t_0}(1)k\}$  is linearly independent.

To minimize  $\|\hat{T}_n^q\|^2$  with respect to  $\hat{\gamma}_i$ ,  $\delta(\|\hat{T}_n^q\|^2)$ , following a change  $\delta(\hat{\gamma}_i)$ , in  $\hat{\gamma}_i$ ,  $i=q+\delta_{t_0}-1(1)k$ , is to vanish.

Proceeding as in the previous theorem, we get the following system of linear equations.

$$\begin{aligned} & h^2 \sum_{i=q+\delta_{t_0}-1}^k \hat{\gamma}_i (D2[t, \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}], D2[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k-\delta_{t_0}}]) \\ &= \sum_{i=0}^k \alpha_i (D2[t, \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}], K(t, \bar{x}_{n+k-1})) \\ &- h^2 \sum_{i=\delta_{t_0}}^{q+\delta_{t_0}-2} \gamma_i (D2[t, \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}], D2[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k-\delta_{t_0}}]), \\ & \qquad \qquad \qquad j = q+\delta_{t_0}-1(1)k. \end{aligned}$$

Thus we get a system of equations

$$(21) \qquad \qquad \qquad Px = Q,$$

where  $P = (P_{ij})_{i,j=q+\delta_{t_0}-1}^k$  is a matrix with

$$\begin{aligned} P_{ij} &= (D2[t, \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}], D2[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k-\delta_{t_0}}]) \\ &= (D2'' [x_{n+k-1}, \dots, x_{n+k-\delta_{t_0}}; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}]), \end{aligned}$$

$$x = [\hat{\gamma}_{q+\delta_{t_0}-1}, \hat{\gamma}_{q+\delta_{t_0}}, \dots, \hat{\gamma}_k]^T,$$

$$Q = [q_{q+\delta_{t_0}-1}, q_{q+\delta_{t_0}}, \dots, q_k]^T \text{ with}$$

$$q_i = \frac{1}{h^2} \left( \sum_{l=0}^k \alpha_l (D2[x_{n+k-l}; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}]) \right. \\ \left. - h^2 \sum_{l=\delta_{t_0}}^{q+\delta_{t_0}-2} \gamma_l (D2''[x_{n+k-l}, \dots, x_{n+k-\delta_{t_0}}; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}]) \right),$$

$$i = q+\delta_{t_0}-1(1)k.$$

The matrix  $P$  can be obtained by performing elementary row and column operations on a leading submatrix of the matrix of the linear system (11). Therefore  $P$  must be nonsingular. So the system (21) can be solved uniquely, and the method (20) can be determined uniquely. By (20) and (21), we get the following theorem.

Theorem 5: The multistep method

$$\sum_{l=0}^k \alpha_l y_{n+k-l} - h^2 \sum_{l=\delta_{t_0}}^k \beta_l f_{n+k-l} = 0$$

is interpolatory for polynomials of degree  $q < k+\delta_{t_1}+1$ , and optimal with respect to  $\beta$ -coefficients if and only if it is interpolatory for the functions

$$\{x^i, i = 0(1)q\} \cup \{h_j(x), j=q+\delta_{t_0}-1(1)k\},$$

where  $h_j(x) = D2[x; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}]$ .

Now we shall find the  $\alpha$ -optimal multistep methods which are interpolatory for polynomials upto degree  $q$ .

If  $m$  is the degree of the highest polynomial precision method corresponding to a multistep method of the form (2) with given  $\beta_i$ 's, then the  $\alpha$ -optimal methods can be made interpolatory for polynomials of degree  $q$  less than  $m$ .

The usual multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t0}}^k \beta_i f_{n+k-i} = 0$$

can be written, using divided difference notation, as

$$\sum_{i=0}^k \gamma_i y[x_{n+k-i}, x_{n+k-i+1}, \dots, x_{n+k}] - h^2 \sum_{i=\delta_{t0}}^k \beta_i f_{n+k-i} = 0.$$

or,

$$(22) \quad \sum_{i=0}^q \gamma_i y[x_{n+k-i}, x_{n+k-i+1}, \dots, x_{n+k}] - h^2 \sum_{i=\delta_{t0}}^k \beta_i f_{n+k-i} \\ = - h^2 \sum_{i=q+1}^k \gamma_i y[x_{n+k-i}, x_{n+k-i+1}, \dots, x_{n+k}].$$

If  $q \leq m-1$ , we can choose the coefficients  $\{\gamma_i, i = 0(1)q\}$  in such a manner that the method is exact for polynomials upto degree  $q$ . In such a method, optimization can be done with respect to the coefficients  $\{\gamma_i, i=q+1(1)k\}$ . So we shall minimize the norm of the local truncation error functional with respect to these coefficients.

Now the divided difference  $y[x_{n+k-i}, x_{n+k-i+1}, \dots, x_{n+k}]$  being a linear combination of  $y(x_{n+k-i}), y(x_{n+k-i+1}), \dots, y(x_{n+k})$ , its representer in  $H$  is  $K[t; \bar{x}_{n+k-i}, \bar{x}_{n+k-i+1}, \dots, \bar{x}_{n+k}]$ , which can be obtained as a divided difference of  $K(t, \bar{x})$  with respect to its 2nd variable, which is a linear combination of  $K(t, \bar{x}_{n+k-i}), \dots, K(t, \bar{x}_{n+k})$ .

Then we have for any function  $y(x) \in H$ ,

$$y[x_{n+k-1}, \dots, x_{n+k}] = (y(t), K[t; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}]).$$

The representer for the local truncation error functional  $\hat{T}_n^q$  of the optimal method, in a Hilbert space possessing a reproducing kernel function is given by

$$(23) \quad \sum_{i=0}^q \bar{\gamma}_i K[t, \bar{x}_{n+k-1}, \bar{x}_{n+k-1+1}, \dots, \bar{x}_{n+k}] - h^2 \sum_{i=\delta_{t_0}}^k \bar{\beta}_i D2(t, \bar{x}_{n+k-1}) \\ + h^2 \sum_{i=q+1}^k \bar{\gamma}_i K[t, \bar{x}_{n+k-1}, \bar{x}_{n+k-1+1}, \dots, \bar{x}_{n+k}].$$

By Lemma 1,  $S_1 = \{K[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}], D2(t, \bar{x}_{n+k-1}) : i = 0(1)k\}$  is linearly independent. To minimize  $\|\hat{T}_n^q\|^2$  with respect to  $\hat{\gamma}_i$ ,  $\delta(\|\hat{T}_n^q\|^2)$ , following a change  $\delta(\hat{\gamma}_i)$ , in  $\hat{\gamma}_i$ ,  $i = q+1(1)k$ , is to vanish. Proceeding as in Theorem 5, we get the following system of linear equations

$$- \sum_{i=q+1}^k \hat{\gamma}_i (K[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}], K[t, \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k}]) \\ = \sum_{i=0}^q \gamma_i (K[t, \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}], K[t, \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k}]) \\ - h^2 \sum_{i=\delta_{t_0}}^k \beta_i (D2(t, \bar{x}_{n+k-1}), K[t, \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k}]), \quad j=q+1(1)k.$$

which may be rewritten as

$$(24) \quad \tilde{P} \tilde{x} = \tilde{Q},$$

where  $\tilde{P} = (\tilde{P}_{ij})_{i,j=q+1}^k$  is a matrix with

$$\tilde{P}_{1j} = K[x_{n+k-j}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}],$$

$$\tilde{x} = [\hat{\gamma}_{q+1}, \hat{\gamma}_{q+2}, \dots, \hat{\gamma}_k]^T,$$

$$\tilde{Q} = [\tilde{q}_{q+1}, \tilde{q}_{q+2}, \dots, \tilde{q}_k]^T \text{ with}$$

$$\begin{aligned} \tilde{q}_1 = & - \sum_{i=0}^q \gamma_i K[x_{n+k-j}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] \\ & + h^2 \sum_{i=\delta_{t0}}^k \beta_i D2(x_{n+k-j}, \dots, x_{n+k}; \bar{x}_{n+k-1}), \quad j=q+1(1)k, \end{aligned}$$

where  $D2(x_{n+k-j}, \dots, x_{n+k}; \bar{x}_{n+k-1})$  is the divided difference of  $D2(t, \bar{z})$  with respect to its first variable. The matrix  $\tilde{P}$  can be obtained by performing elementary row and column operations on a leading submatrix of the matrix of the linear system (13). Therefore  $\tilde{P}$  must be nonsingular. So the system (24) can be solved uniquely, and the method (23) can be determined uniquely. By (23) and (24), we get the following theorem.

Theorem 6: The multistep method

$$\sum_{i=0}^k \hat{\alpha}_i y_{n+k-i} - h^2 \sum_{i=\delta_{t0}}^k \beta_i f_{n+k-i} = 0$$

is interpolatory for polynomials of degree  $q \leq m-1$ , and optimal with respect to  $\alpha$ -coefficients if and only if it is interpolatory for the functions

$$\{x^i, i = 0(1)q\} \cup \{h_j(x), j = q+1(1)k\},$$

where  $h_j(x) = K[x_{n+k-j}, \dots, x_{n+k}; \bar{x}]$ .

Now we shall find the  $\alpha$  as well as  $\beta$ -optimal multistep methods which is interpolatory for polynomials upto degree  $q$ .

The usual multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0$$

can be written, using divided difference notation, as

$$\sum_{i=0}^k \lambda_i y[x_{n+k-i}, \dots, x_{n+k}] - h^2 \sum_{i=\delta_{t_0}}^k \mu_i y''[x_{n+k-i}, \dots, x_{n+k}] = 0.$$

We can choose the coefficients  $\{\lambda_i, i=0(1)q\}$  and  $\{\mu_i, i=\delta_{t_0}(1)q+\delta_{t_0}-2\}$  in such a manner that the method is exact for polynomials upto degree  $q$ . Thus we have

$$\begin{aligned} & \sum_{i=0}^q \lambda_i y[x_{n+k-i}, \dots, x_{n+k}] - h^2 \sum_{i=\delta_{t_0}}^{q+\delta_{t_0}-2} \mu_i y''[x_{n+k-i}, \dots, x_{n+k}] \\ &= - \sum_{i=q+1}^k \lambda_i y[x_{n+k-i}, \dots, x_{n+k}] + h^2 \sum_{i=q+\delta_{t_0}-1}^k \mu_i y''[x_{n+k-i}, \dots, x_{n+k}], \end{aligned}$$

in which optimization can be done with respect to the coefficients  $\{\lambda_i, i = q+1(1)k\}$  and  $\{\mu_i, i = q+\delta_{t_0}-2(1)k\}$ . So we shall minimize the norm of the local truncation error functional with respect to these coefficients. The representer for the local truncation error functional  $\hat{T}_n^q$  of the optimal method, in a Hilbert space possessing a reproducing kernel function is given by

$$\sum_{i=0}^q \bar{\lambda}_i K[t; \bar{x}_{n+k-i}, \dots, \bar{x}_{n+k}] - h^2 \sum_{i=\delta_{t_0}}^{q+\delta_{t_0}-2} \bar{\mu}_i D_2[t; \bar{x}_{n+k-i}, \dots, \bar{x}_{n+k}]$$

$$= - \sum_{i=q+1}^k \bar{\lambda}_i K[t; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] + h^2 \sum_{i=q+\delta_{t_0}-1}^k \bar{\mu}_i D2[t; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}]$$

By Lemma 1,  $S_1 = \{K[t; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}], D2[t; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}]: i=0(1)k\}$  is linearly independent. To minimize  $\|\hat{T}_n^q\|^2$  with respect to  $\hat{\lambda}_i$  and  $\hat{\mu}_i$ ,  $\delta(\|\hat{T}_n^q\|^2)$  following a change  $\delta(\hat{\lambda}_i)$  in  $\hat{\lambda}_i$ ,  $i=q+1(1)k$ , and  $\delta(\|\hat{T}_n^q\|^2)$ , following a change  $\delta(\hat{\mu}_i)$ , in  $\hat{\mu}_i$ ,  $i=q+\delta_{t_0}-1(1)k$  are to vanish. Proceeding as in Theorem 5, we get the normal equations,

$$\begin{aligned} (25) \quad & - \sum_{i=q+1}^k \hat{\lambda}_i K[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] \\ & + h^2 \sum_{i=q+\delta_{t_0}-1}^k \hat{\mu}_i D2[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] \\ & = \sum_{i=0}^q \lambda_i K[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] \\ & - h^2 \sum_{i=\delta_{t_0}}^{q+\delta_{t_0}-2} \mu_i D2[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}], \\ & \qquad \qquad \qquad l = q+1(1)k, \end{aligned}$$

and

$$\begin{aligned} (26) \quad & - \sum_{i=q+1}^k \hat{\lambda}_i D2[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] \\ & + h^2 \sum_{i=q+\delta_{t_0}-1}^k \hat{\mu}_i D2''[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] \\ & = \sum_{i=0}^q \lambda_i D2[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}] \end{aligned}$$

$$- h^2 \sum_{l=\delta_{t_0}}^{q+\delta_{t_0}-2} \mu_l D2'' [x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}],$$

$$l = q+\delta_{t_0}-1(1)k.$$

Thus we get a system of linear equations, written in matrix form as

$$(27) \quad \begin{bmatrix} A & B \\ -B^* & C \end{bmatrix} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

where

$$A = (A_{lj})_{\substack{l=q+1(1)k \\ j=q+1(1)k}}, \quad A_{lj} = -K[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k}],$$

$$B = (B_{lj})_{\substack{l=q+1(1)k \\ j=q+\delta_{t_0}-1(1)k}}, \quad B_{lj} = D2[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k}],$$

$$C = (C_{lj})_{\substack{l=q+\delta_{t_0}-1(1)k \\ j=q+\delta_{t_0}-1(1)k}}, \quad C_{lj} = D2''[x_{n+k-j}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}],$$

$$\lambda = (\hat{\lambda}_{q+1}, \hat{\lambda}_{q+2}, \dots, \hat{\lambda}_k)^T, \quad \mu = (\hat{\mu}_{q+\delta_{t_0}-1}, \hat{\mu}_{q+\delta_{t_0}}, \dots, \hat{\mu}_k)^T,$$

$$d_1 = (d_{1,q+1}, d_{1,q+2}, \dots, d_{1,k})^T, \text{ with}$$

$$d_{1,l} = \sum_{j=0}^q \lambda_j K[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k}]$$

$$- h^2 \sum_{j=\delta_{t_0}}^{q+\delta_{t_0}-2} \mu_l D2[x_{n+k-1}, \dots, x_{n+k}; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k}], \quad l=q+1(1)k,$$

$$d_2 = (d_{2,q+\delta_{t_0}-1}, d_{2,q+\delta_{t_0}}, \dots, d_{2,k})^T, \text{ with}$$

$$d_{2l} = \sum_{j=0}^q \lambda_j D2[x_{n+k-j}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}]$$



$$- h^2 \sum_{j=\delta_{t_0}}^{q+\delta_{t_0}-2} \mu_j D_2'' [x_{n+k-j}, \dots, x_{n+k}; \bar{x}_{n+k-1}, \dots, \bar{x}_{n+k}], \quad i=q+\delta_{t_0}-1(1)k.$$

Thus we get the following theorem.

Theorem 7: The multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0$$

is interpolatory for polynomials of degree  $q$ , and optimal with respect to  $\alpha$  as well as  $\beta$  coefficients if and only if it is interpolatory for the functions

$$\{x^i, i=0(1)q\} \cup \{h_j(x), j=q+1(1)k\} \cup \{g_j(x), j=q+\delta_{t_0}-1(1)k\}$$

where 
$$h_j(x) = K [x_{n+k-j}, \dots, x_{n+k}; \bar{x}]$$

and 
$$g_j(x) = D_2[x; \bar{x}_{n+k-j}, \dots, \bar{x}_{n+k-\delta_{t_0}}].$$

#### 1.4 Optimal Multistep Methods Interpolatory for Arbitrary Functions

We shall find the optimal multistep methods interpolatory for arbitrary functions with the help of Lagrangian multipliers.

Lemma 2 : A necessary and sufficient condition that the multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0$$

be interpolatory for functions  $f_1, f_2, \dots, f_q$  is that for every function  $\phi \in \text{span} \{f_1, f_2, \dots, f_q\}$  for which

$$\phi(x_{n+k-1}) = 0, \quad i = 0(1)k,$$

there holds

$$\sum_{i=\delta_{t0}}^k \beta_i \phi''(x_{n+k-1}) = 0.$$

Proof : The necessity part is obvious. For sufficiency assume that the stated conditions hold. Consider the system

$$(28) \quad \sum_{j=0}^k \alpha_j f_i(x_{n+k-j}) = h^2 \sum_{j=\delta_{t0}}^k \beta_j f_i''(x_{n+k-j}), \quad 1 \leq i \leq q$$

of linear equations in  $\alpha_j$ ,  $j = 0(1)k$ . Applying elementary row operations on the system (28), we can reduce it to one in which the matrix is in a row reduced echolon form. In this system the existence of a nonzero augmented entry corresponding to a zero row of the matrix would contradict the stated conditions. Hence, the system (28) is consistent, completing the proof of the lemma.

In view of the above lemma and its proof, we can say that if a multistep method is interpolatory for span  $\{f_1, f_2, \dots, f_q\}$  then there exists  $q_1 \leq k+1$  functions  $g_1, g_2, \dots, g_{q_1}$  in the span  $\{f_1, f_2, \dots, f_q\}$  such that

$$\text{rank} [g_i(x_{n+k-j})]_{\substack{i=1(1)q_1 \\ j=0(1)k}} = q_1,$$

and the multistep method is interpolatory for span  $\{f_1, f_2, \dots, f_q\}$  if and only if, it is interpolatory for  $g_1, g_2, \dots, g_{q_1}$ . Hence with preprocessing, if necessary, we can assume that  $f_1, f_2, \dots, f_q$  themselves are such that

$$(29) \quad \text{rank} [f_i(x_{n+k-j})]_{\substack{i=1(1)q \\ j=0(1)k}} = q \leq k+1.$$

If  $q = k+1$ , all the coefficients  $\alpha_i$ 's get uniquely determined and no optimization of error will be possible. Hence for  $\alpha$ -optimal interpolatory method, we will assume that  $q < k+1$ .

Lemma 3 : A necessary and a sufficient condition that the multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0$$

be interpolatory for arbitrary functions  $f_1, f_2, \dots, f_q$  is that for every function  $\phi \in \text{span} \{f_1, f_2, \dots, f_q\}$  for which

$$\phi''(x_{n+k-i}) = 0, \quad i = \delta_{t_0}(1)k,$$

there holds

$$\sum_{i=0}^k \alpha_i \phi(x_{n+k-i}) = 0.$$

Proof : Proof follows similar to the proof of Lemma 2.

With the similar lines of arguments, we see that if a multistep method is interpolatory for  $\text{span} \{f_1, f_2, \dots, f_q\}$ , then there exists  $q_1 (\leq k + \delta_{t_1})$  functions  $g_1, g_2, \dots, g_{q_1}$  in the  $\text{span} \{f_1, f_2, \dots, f_q\}$  such that

$$\text{rank} [g_i''(x_{n+k-j})]_{\substack{i=1(1)q_1 \\ j=\delta_{t_0}(1)k}} = q_1,$$

and the multistep method is interpolatory for  $\text{span} \{f_1, f_2, \dots, f_q\}$  if and only if, it is interpolatory for  $g_1, g_2, \dots, g_{q_1}$ . Hence we can assume that  $f_1, f_2, \dots, f_q$  themselves are such that

$$(30) \quad \text{rank} [f''_i(x_{n+k-j})]_{\substack{i=1(1)q \\ j=\delta_{t_0}(1)k}} = q \leq k + \delta_{t_1}.$$

If  $q = k + \delta_{t_1}$ , all the coefficients  $\beta_i$ 's get uniquely determined and no optimization of error will be possible. Hence for  $\beta$ -optimal interpolatory method, we will assume that  $q < k + \delta_{t_1}$ .

Lemma 4 : A necessary and sufficient condition that the multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^{k-1} \beta_i (f_{n+k-i} - f_n) - h^2 f_n = 0$$

be interpolatory for linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$  is that for every function  $\phi \in \text{span} \{f_1, f_2, \dots, f_q\}$  for which

$$\phi''(x_{n+k-i}) - \phi''(x_n) = 0, \quad i = \delta_{t_0}(1)k-1,$$

there holds

$$\sum_{i=0}^k \alpha_i \phi(x_{n+k-i}) - h^2 \phi''(x_n) = 0.$$

Proof : The necessity part is obvious. For sufficiency, assume that the stated conditions hold. Consider the system

$$(31) \quad h^2 \sum_{j=\delta_{t_0}}^{k-1} \beta_j \left( f''_i(x_{n+k-j}) - f''_i(x_n) \right) = \sum_{j=0}^k \alpha_j f_i(x_{n+k-j}) - h^2 f''_i(x_n),$$

$$1 \leq i \leq q,$$

of linear equations in  $\beta_j$ ,  $j = \delta_{t_0}(1)k-1$ . With the similar arguments given in Lemma 2, the sufficiency part follows.

Thus, we can say that if a multistep method is interpolatory

for span  $\{f_1, f_2, \dots, f_q\}$ , then there exists  $q_1 \leq k-1+\delta_{t_1}$  functions  $g_1, g_2, \dots, g_{q_1}$  in the span  $\{f_1, f_2, \dots, f_q\}$ , such that

$$\text{rank} \left( g_1''(x_{n+k-j}) - g_1''(x_n) \right)_{\substack{i=1(1)q_1 \\ j=\delta_{t_0}(1)k-1}} = q_1,$$

and the multistep method is interpolatory for span  $\{f_1, f_2, \dots, f_q\}$  if and only if, it is interpolatory for  $g_1, g_2, \dots, g_{q_1}$ . So, we can assume that  $f_1, f_2, \dots, f_q$  themselves are such that

$$(32) \quad \text{rank} \left( f_1''(x_{n+k-j}) - f_1''(x_n) \right)_{\substack{i=1(1)q \\ j=\delta_{t_0}(1)k-1}} = q \leq k-1+\delta_{t_1}.$$

If  $q = k-1+\delta_{t_1}$ , all the coefficients  $\beta_i$ 's get uniquely determined and no optimization of error will be possible. Hence for optimal interpolatory method with restriction we will assume that  $q < k-1+\delta_{t_1}$ .

First, we shall find  $\beta$ -optimal multistep method interpolatory for a set of arbitrary functions. Let

$$(33) \quad \sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F f_{n+k-i} = 0$$

be the  $\beta$ -optimal method interpolatory for functions  $f_1, f_2, \dots, f_q$ . The representer for the local truncation error functional in a Hilbert space possessing a reproducing kernel function is given by

$$\hat{T}_n^F = \sum_{i=0}^k \bar{\alpha}_i K(t, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^k \bar{\beta}_{i,n}^F D_2(t, \bar{x}_{n+k-i}).$$

To obtain the optimal coefficients  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k$ , we have to minimize,  $\|\hat{T}_n^F\|^2$  subject to the condition,  $(\hat{T}_n^F, f_m) = 0$ ,  $m = 1(1)q$ . Using the method of Lagrange multipliers, we get the following

system of linear equations to determine  $\hat{\beta}_{1,n}^F$ ,  $i = \delta_{t_0}(1)k$ .

$$(34) \quad \begin{bmatrix} A & -F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} b \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where

$$A = (A_{ij})_{i=\delta_{t_0}(1)k, j=\delta_{t_0}(1)k}, \quad \text{with} \quad A_{ij} = D2''(x_{n+k-j}, \bar{x}_{n+k-1}),$$

$$F = (F_{ij})_{i=1(1)q, j=\delta_{t_0}(1)k}, \quad \text{with} \quad F_{ij} = f_i''(x_{n+k-j}),$$

$$b = h^2 \left( \hat{\beta}_{\delta_{t_0}, n}^F, \hat{\beta}_{\delta_{t_0}+1, n}^F, \dots, \hat{\beta}_{k, n}^F \right)^T, \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T,$$

$$c = \left( c_{\delta_{t_0}}, c_{\delta_{t_0}+1}, \dots, c_k \right)^T, \quad \text{with} \quad c_i = \sum_{j=0}^k \alpha_j D2(x_{n+k-j}, \bar{x}_{n+k-1}),$$

and

$$f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T, \quad \text{with} \quad \tilde{f}_i = \sum_{j=0}^k \alpha_j f_i(x_{n+k-j}).$$

Assuming the functions  $f_m$ ,  $m = 1(1)q$ , to be such that the above matrix is invertible, it follows that the unique solution  $\hat{\beta}_{1,n}^F$ ,  $i = \delta_{t_0}(1)k$ , thus obtained will satisfy the required optimal interpolatory condition.

Now we shall characterize the  $\beta$ -optimal multistep method interpolatory for a set of functions. For simplicity in the form of characterization, we assume that the last  $p$  columns in the matrix (30) are linearly independent. In general, our assumption is that some  $p$  columns of the matrix (30) are linearly independent. Then in the corresponding characterization, the indices of these columns instead of the last  $p$  columns in the simpler case would appear in the formulation.

Let the interpolatory condition appearing in (34),

$$Fb = f$$

be rewritten as

$$[E \mid P] \begin{bmatrix} d \\ e \end{bmatrix} = f,$$

where

$$(35) \quad E = \left( f''_1(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=\delta_{t0}(1)k-q}}^{\substack{i=1(1)q \\ j=k-q+1(1)k}}, \quad P = \left( f''_1(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=k-q+1(1)k}},$$

$$(36) \quad d = \left( \hat{\beta}_{\delta_{t0}, n}^F, \dots, \hat{\beta}_{k-q, n}^F \right)^T, \quad e = \left( \hat{\beta}_{k-q+1, n}^F, \dots, \hat{\beta}_{k, n}^F \right)^T,$$

$$(37) \quad f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T, \quad \text{with } \tilde{f}_1 = \frac{1}{h^2} \sum_{j=0}^k \alpha_j f_1(x_{n+k-j}).$$

Then,  $e = P^{-1}(f - Ed)$ . Let

$$(38) \quad r = P^{-1}f = (r_1, r_2, \dots, r_q)^T, \quad G = P^{-1}E = (g_{ij})_{\substack{i=1(1)q \\ j=1(1)k-q-\delta_{t0}+1}},$$

then  $e = r - Gd$ , and the method (33) can be written as

$$\sum_{l=0}^k \alpha_l y_{n+k-l} - h^2 \sum_{l=\delta_{t0}}^{k-q} \hat{\beta}_{l, n}^F f_{n+k-l} - h^2 \sum_{l=k-q+1}^k \hat{\beta}_{l, n}^F f_{n+k-l} = 0,$$

or,

$$\begin{aligned} & \sum_{l=0}^k \alpha_l y_{n+k-l} - h^2 \sum_{l=\delta_{t0}}^{k-q} \hat{\beta}_{l, n}^F f_{n+k-l} \\ & - h^2 \sum_{l=k-q+1}^k \left( r_{l+q-k} - \sum_{j=\delta_{t0}}^{k-q} \hat{\beta}_{j, n}^F g_{l+q-k, j+1-\delta_{t0}} \right) f_{n+k-l} = 0, \end{aligned}$$

which is the same as

$$(39) \quad \sum_{l=0}^k \alpha_l y_{n+k-l} - h^2 \sum_{l=k-q+1}^k r_{l+q-k} f_{n+k-l} -$$

$$- h^2 \sum_{i=\delta_{t_0}}^{k-q} \hat{\beta}_{i,n}^F \left( f_{n+k-i} - \sum_{j=k-q+1}^k g_{j+q-k, i+1-\delta_{t_0}} f_{n+k-j} \right) = 0.$$

Hence we get the following Theorem:

Theorem 8: An optimal multistep method

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F f_{n+k-i} = 0$$

with prefixed  $\alpha_i$ 's, is interpolatory for functions  $f_1, f_2, \dots, f_q$  satisfying (30), if and only if it is interpolatory for the functions

$$\{f_1, f_2, \dots, f_q\} \cup \{h_i : i = \delta_{t_0}(1)k-q\},$$

$$\text{where } h_i = D2(x, \bar{x}_{n+k-i}) - \sum_{j=k-q+1}^k \bar{g}_{j+q-k, i+1-\delta_{t_0}} D2(x, \bar{x}_{n+k-j})$$

where  $g_{ij}$ 's are defined in (38) and (35).

Proof: From equation (39), we see that to determine the optimal coefficients  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k-q$ , the normal equations can be obtained by minimizing the norm of local truncation error functional  $\hat{T}_n^F$  for the method (39) with respect to the coefficients  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k-q$ . Thus we get the system of normal equations

$$(h_i, \hat{T}_n^F) = 0, \quad i = \delta_{t_0}(1)k-q,$$

$$\text{where } h_i = D2(x, \bar{x}_{n+k-i}) - \sum_{j=k-q+1}^k \bar{g}_{j+q-k, i+1-\delta_{t_0}} D2(x, \bar{x}_{n+k-j}) \text{ and the}$$

method (39) is also interpolatory for the functions  $f_1, f_2, \dots, f_q$ .

Hence the proof.



Now we shall find an  $\alpha$ -optimal method interpolatory for linearly independent arbitrary functions.

Let

$$(40) \quad \sum_{i=0}^k \hat{\alpha}_{i,n}^F y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0$$

be an  $\alpha$ -optimal method interpolatory for arbitrary functions  $f_1, f_2, \dots, f_q$ . The representer for the local truncation error functional in a Hilbert space possessing a reproducing kernel function is given by

$$\hat{T}_n^F = \sum_{i=0}^k \bar{\alpha}_{i,n}^F K(t, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^k \bar{\beta}_i D_2(t, \bar{x}_{n+k-i})$$

To obtain the optimal coefficients  $\hat{\alpha}_{i,n}^F$ ,  $i = 0(1)k$ , we have to minimize,  $\|\hat{T}_n^F\|^2$  subject to the condition  $(\hat{T}_n^F, f_m) = 0$ ,  $m = 1(1)q$ . Using method of Lagrange multipliers, we get the following system of linear equations to determine  $\hat{\alpha}_{i,n}^F$ ,  $i = 0(1)k$ .

$$(41) \quad \begin{bmatrix} A & F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} a \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where  $F^*$  is the conjugate transpose of  $F$ .

$$A = (A_{ij})_{i=0(1)k, j=0(1)k}, \quad \text{with} \quad A_{ij} = K(x_{n+k-j}, \bar{x}_{n+k-i}),$$

$$F = (F_{ij})_{i=1(1)q, j=0(1)k}, \quad \text{with} \quad F_{ij} = f_i(x_{n+k-j}),$$

$$a = (\hat{\alpha}_{0,n}^F, \hat{\alpha}_{1,n}^F, \dots, \hat{\alpha}_{k,n}^F)^T, \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T,$$

$$c = (c_0, c_1, \dots, c_k)^T, \quad \text{with } c_i = h^2 \sum_{j=\delta_{t0}}^k \beta_j D_2(x_{n+k-1}, \bar{x}_{n+k-j}),$$

and

$$f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T, \quad \text{with } \tilde{f}_i = h^2 \sum_{j=\delta_{t0}}^k \beta_j f''_i(x_{n+k-j}).$$

Assuming the functions  $\tilde{f}_m$ 's to be such that the above matrix is invertible,  $\hat{\alpha}_{i,n}^F$ ,  $i = 0(1)k$ , can be determined uniquely.

Now we shall characterize the  $\alpha$ -optimal multistep method interpolatory for a given set of functions. For simplicity in the form of characterization, we assume that the last  $p$  columns in the matrix (29) are linearly independent. In general, our assumption is that some  $p$  columns of the matrix (29) are linearly independent. Then in the corresponding characterization, the indices of these columns instead of the last  $p$  columns in the simpler case would appear in the formulation.

Let the interpolatory condition appearing in (41),

$$Fa = f$$

be rewritten as

$$[E \mid P] \begin{bmatrix} d \\ e \end{bmatrix} = f,$$

where

$$(42) \quad E = \left( f_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=0(1)k-q}}, \quad P = \left( f_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=k-q+1(1)k}},$$

$$(43) \quad d = \left( \hat{\alpha}_{0,n}^F, \dots, \hat{\alpha}_{k-p,n}^F \right)^T, \quad e = \left( \hat{\alpha}_{k-p+1,n}^F, \dots, \hat{\alpha}_{k,n}^F \right)^T$$

$$(44) \quad f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T, \quad \text{with } \tilde{f}_i = h^2 \sum_{j=\delta_{t_0}}^k \beta_j f''_i(x_{n+k-j}).$$

Then,  $e = P^{-1}(f - Ed)$ . Let

$$(45) \quad r = P^{-1}f = (r_1, r_2, \dots, r_q)^T, \quad G = P^{-1}E = (g_{ij})_{\substack{i=1(1)q \\ j=1(1)k-q-\delta_{t_0}+1}},$$

then  $e = r - Gd$ . Then the method (40) can be written as

$$\sum_{i=0}^{k-q} \hat{\alpha}_{i,n}^F y_{n+k-i} + \sum_{i=k-q+1}^k \hat{\alpha}_{i,n}^F y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0,$$

or,

$$\begin{aligned} \sum_{i=0}^{k-q} \hat{\alpha}_{i,n}^F y_{n+k-i} + \sum_{i=k-q+1}^k \left( r_{i+q-k} - \sum_{j=0}^{k-q} \hat{\alpha}_{j,n}^F g_{i+q-k, j+1} \right) y_{n+k-i} \\ - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0, \end{aligned}$$

or,

$$\begin{aligned} \sum_{i=0}^{k-q} \hat{\alpha}_{i,n}^F y_{n+k-i} + \sum_{j=k-q+1}^k \left( r_{j+q-k} - \sum_{i=0}^{k-q} \hat{\alpha}_{i,n}^F g_{j+q-k, i+1} \right) y_{n+k-j} \\ - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0; \end{aligned}$$

or, we get

$$(46) \quad \begin{aligned} \sum_{i=0}^{k-q} \hat{\alpha}_{i,n}^F \left( y_{n+k-i} - \sum_{j=k-q+1}^k g_{j+q-k, i+1} y_{n+k-j} \right) + \\ \sum_{j=k-q+1}^k r_{j+q-k} y_{n+k-j} - h^2 \sum_{i=\delta_{t_0}}^k \beta_i f_{n+k-i} = 0 \end{aligned}$$

Hence we get the following Theorem:

Theorem 9: An optimal multistep method

$$\sum_{i=0}^k \hat{\alpha}_{i,n}^F Y_{n+k-i} - h^2 \sum_{i=\delta_{t0}}^k \beta_i f_{n+k-i} = 0,$$

where  $\beta_i$ 's are prefixed, is interpolatory for arbitrary functions  $f_1, f_2, \dots, f_q$ , if and only if it is interpolatory for the functions

$$\{f_1, f_2, \dots, f_q\} \cup \{h_i : i = 0(1)k-q\},$$

$$\text{where } h_i = K(x, \bar{x}_{n+k-i}) + \sum_{j=k-q+1}^k \bar{g}_{j+q-k, i+1} K(x, \bar{x}_{n+k-j}),$$

where  $r_i$  and  $g_{ij}$ 's are defined in (42), (44), (45).

Proof: From equation (46) we find that to determine the optimal coefficients  $\hat{\alpha}_{i,n}^F$ ,  $i = 0(1)k-q$ , the normal equations can be obtained by minimizing the norm of the local truncation error functional  $\hat{T}_n^F$  for the method (46), with respect to the coefficients  $\hat{\alpha}_{i,n}^F$ ,  $i = 0(1)k-q$ . Thus we get the system of normal equations

$$(h_i, \hat{T}_n^F) = 0, \quad i = 0(1)k-q$$

$$\text{where } h_i = K(x, \bar{x}_{n+k-i}) + \sum_{j=k-q+1}^k \bar{g}_{j+q-k, i+1} K(x, \bar{x}_{n+k-j}),$$

and the method (46) is also interpolatory for the functions  $f_1, f_2, \dots, f_q$ . Hence the proof.

Now we shall find  $\beta$ -optimal multistep method interpolatory for linearly independent arbitrary functions under some restriction.

Let

$$(47) \quad \sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F f_{n+k-i} = 0$$

be the optimal multistep method interpolatory for linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$ , where  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k$  are the optimal coefficients to be determined subject to the condition that

$$(48) \quad \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F = 1.$$

For convenience, let us write,  $\hat{\beta}_{k,n}^F = 1 - \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}^F$ .

Then the method (47) becomes

$$(48) \quad \sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}^F (f_{n+k-i} - f_n) - h^2 f_n = 0.$$

The representer for the local truncation error functional for the method (48), in a Hilbert space possessing a reproducing kernel function is given by

$$\hat{T}_n^F = \sum_{i=0}^k \bar{\alpha}_i K(t, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^{k-1} \bar{\beta}_{i,n}^F \left( D2(t, \bar{x}_{n+k-i}) - D2(t, \bar{x}_n) \right) - h^2 D2(t, \bar{x}_n).$$

To obtain the optimal coefficients  $\hat{\beta}_{i,n}^F$ ,  $i=\delta_{t_0}(1)k-1$ , we have to minimize,  $\|\hat{T}_n^F\|^2$  subject to the condition  $(\hat{T}_n^F, f_m) = 0$ ,  $m = 1(1)q$ . Using the methodology of Lagrange multipliers, we get the following system of equations to determine  $\hat{\beta}_{i,n}^F$ ,  $i=\delta_{t_0}(1)k-1$ .

$$(49) \quad \begin{bmatrix} A & -F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} b \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where  $F^*$  is the complex conjugate of  $F$ .

$$A = (A_{ij})_{i=\delta_{t_0}(1)k-1, j=\delta_{t_0}(1)k-1}, \quad \text{with}$$

$$A_{ij} = D2''(x_{n+k-j}, \bar{x}_{n+k-i}) - D2''(x_{n+k-j}, \bar{x}_n) - D2''(x_n, \bar{x}_{n+k-i}) + D2''(x_n, \bar{x}_n),$$

$$F = (F_{ij})_{i=1(1)q, j=\delta_{t_0}(1)k-1}, \quad \text{with } F_{ij} = f_i''(x_{n+k-j}) - f_i''(x_n),$$

$$b = h^2 \left( \hat{\beta}_{\delta_{t_0}, n}^F, \hat{\beta}_{\delta_{t_0}+1, n}^F, \dots, \hat{\beta}_{k-1, n}^F \right)^T, \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T,$$

$$c = \left( c_{\delta_{t_0}}, c_{\delta_{t_0}+1}, \dots, c_{k-1} \right)^T, \quad \text{with}$$

$$c_i = \sum_{j=0}^k \alpha_j \left( D2(x_{n+k-j}, \bar{x}_{n+k-i}) - D2(x_{n+k-j}, \bar{x}_n) \right) - h^2 \left( D2''(x_n, \bar{x}_{n+k-i}) - D2''(x_n, \bar{x}_n) \right), \quad \text{for } i = \delta_{t_0}(1)k-1,$$

and

$$f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T, \quad \text{with } \tilde{f}_i = \sum_{j=0}^k \alpha_j f_i(x_{n+k-j}) - h^2 f_i(x_n).$$

If the above matrix is invertible, then the system of linear equations can be solved to have the unique solution,  $\hat{\beta}_{i, n}^F$ ,  $i=\delta_{t_0}(1)k-1$ .

Now we shall characterize the  $\beta$ -optimal multistep method interpolatory for arbitrary functions under the restriction (48). For simplicity in the form of characterization, we assume that the last  $p$  columns in the matrix (32) are linearly independent. In general, our assumption is that some  $p$  columns of the matrix (32) are linearly independent. Then in the corresponding

characterization, the indices of these columns instead of the last  $p$  columns in the simpler case would appear in the formulation.

Let the interpolatory condition appearing in (49)

$$Fb = f,$$

be rewritten as

$$[E \mid P] \begin{bmatrix} d \\ e \end{bmatrix} = f,$$

where

$$(50) \quad E = \left( f''_i(x_{n+k-j}) - f''_i(x_n) \right)_{\substack{i=1(1)q \\ j=\delta_{t0}(1)k-q-1}},$$

$$(51) \quad P = \left( f''_i(x_{n+k-j}) - f''_i(x_n) \right)_{\substack{i=1(1)q \\ j=k-q(1)k-1}},$$

$$(52) \quad d = \left( \hat{\beta}_{\delta_{t0}, n}^F, \dots, \hat{\beta}_{k-q-1, n}^F \right)^T, \quad e = \left( \hat{\beta}_{k-q, n}^F, \dots, \hat{\beta}_{k-1, n}^F \right)^T,$$

$$(53) \quad f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T, \text{ with}$$

$$\tilde{f}_1 = \frac{1}{h^2} \left( \sum_{j=0}^k \alpha_j f_1(x_{n+k-j}) - h^2 f_1(x_n) \right).$$

Then  $e = P^{-1}(f - Ed)$ . Let

$$(54) \quad r = P^{-1}f = (r_0, r_1, \dots, r_{q-1})^T,$$

$$(55) \quad G = P^{-1}E = (g_{ij})_{\substack{i=0(1)q-1 \\ j=1(1)k-q-\delta_{t0}}};$$

then  $e = r - Gd$ . Then the method (48) can be written as

$$\sum_{i=0}^k \alpha_i y_{n+k-i} - h^2 \sum_{i=\delta_{t0}}^{k-q-1} \hat{\beta}_{i,n}^F (f_{n+k-i} - f_n) - h^2 \sum_{i=k-q}^{k-1} \hat{\beta}_{i,n}^F (f_{n+k-i} - f_n) - h^2 f_n = 0,$$

$$\text{or, } \sum_{i=0}^k \alpha_i Y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^{k-q-1} \hat{\beta}_{i,n}^F (f_{n+k-i} - f_n) - h^2 f_n \\ - h^2 \sum_{i=k-q}^{k-1} \left( r_{i+q-k} - \sum_{j=\delta_{t_0}}^{k-q-1} \hat{\beta}_{j,n}^F g_{i+q-k, j+1-\delta_{t_0}} \right) (f_{n+k-i} - f_n) = 0,$$

or,

$$(56) \quad \sum_{i=0}^k \alpha_i Y_{n+k-i} - h^2 \sum_{i=k-q}^{k-1} r_{i+q-k} f_{n+k-i} - h^2 f_n \\ - h^2 \sum_{i=\delta_{t_0}}^{k-q-1} \hat{\beta}_{i,n}^F \left( (f_{n+k-i} - f_n) - \sum_{j=k-q}^{k-1} g_{j+q-k, i+1-\delta_{t_0}} f_{n+k-j} \right) = 0.$$

Hence we get the following Theorem:

Theorem 10: The optimal multistep method,

$$\sum_{i=0}^k \alpha_i Y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F f_{n+k-i} = 0,$$

where  $\alpha_i$ 's are prefixed and  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k$  are the optimal coefficients to be determined subject to the condition

$$\sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F = 1,$$

is interpolatory for arbitrary functions  $f_1, f_2, \dots, f_q$ ; if and only if it is interpolatory for the functions

$$\{f_1, f_2, \dots, f_q\} \cup \{h_i : i = \delta_{t_0}(1)k-q-1\},$$

where  $h_i = D_2(x, \bar{x}_{n+k-i}) - D_2(x, \bar{x}_n)$



$$- \sum_{j=k-q}^{k-1} \bar{g}_{j+q-k, i+1-\delta_{t_0}} \left( D2(x, \bar{x}_{n+k-j}) - D2(x, \bar{x}_n) \right),$$

with  $g_{ij}$ 's defined by (55), (50), (51).

Proof: From equation (56), we see that to determine the optimal coefficients  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k-q-1$  the normal equations can be obtained by minimizing the norm of the local truncation error functional  $\hat{T}_n^F$  for the method (56), with respect to the coefficients  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k-q-1$ . Thus we get the system of normal equations

$$(h_i, \hat{T}_n^F) = 0, \quad i = \delta_{t_0}(1)k-q-1,$$

where  $h_i(x) = D2(x, \bar{x}_{n+k-1}) - D2(x, \bar{x}_n)$

$$- \sum_{j=k-q}^{k-1} \left( \bar{g}_{j+q-k, i+1-\delta_{t_0}} D2(x, \bar{x}_{n+k-j}) - D2(x, \bar{x}_n) \right),$$

and the method (56) is also interpolatory for the functions  $f_1, f_2, \dots, f_q$ . Hence the proof.

Now we shall find  $\alpha$  as well as  $\beta$ -optimal multistep method interpolatory for linearly independent arbitrary functions under some restriction.

Let

$$(57) \quad \sum_{i=0}^k \hat{\alpha}_{i,n}^F y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F f_{n+k-i} = 0$$

be the optimal method interpolatory for arbitrary functions  $f_1, f_2, \dots, f_q$ , where  $\hat{\alpha}_{i,n}^F$ ,  $i = 0(1)k$  and  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t_0}(1)k$  are the optimal coefficients to be determined subject to the condition

$$(58) \quad \sum_{i=\delta_{t_0}}^k \hat{\beta}_{i,n}^F = 1.$$

For convenience, let us write

$$\hat{\beta}_{k,n}^F = 1 - \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}^F.$$

Then the method (57) becomes

$$(59) \quad \sum_{i=0}^k \hat{\alpha}_{i,n}^F y_{n+k-i} - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}^F (f_{n+k-i} - f_n) - h^2 f_n = 0.$$

The representer for the local truncation error functional for the method (59) in a Hilbert space possessing a reproducing kernel function is given by

$$\hat{T}_n^F = \sum_{i=0}^k \hat{\alpha}_{i,n}^F K(t, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}^F \left( D_2(t, \bar{x}_{n+k-i}) - D_2(t, \bar{x}_n) \right) - h^2 D_2(t, \bar{x}_n).$$

To obtain the optimal coefficients  $\hat{\alpha}_{i,n}^F$ ,  $i=0(1)k$  and  $\hat{\beta}_{i,n}^F$ ,  $i=\delta_{t_0}(1)k-1$ , at a point  $x_n$ , we have to minimize,  $\|\hat{T}_n^F\|^2$  subject to the condition  $(\hat{T}_n^F, f_m) = 0$ ,  $m = 1(1)q$ . Thus we get the following system of linear equations

$$(60) \quad \sum_{i=0}^k \hat{\alpha}_{i,n}^F K(x_{n+k-i}, \bar{x}_{n+k-i}) - h^2 \sum_{i=\delta_{t_0}}^{k-1} \hat{\beta}_{i,n}^F \left( \overline{D_2(x_{n+k-i}, \bar{x}_{n+k-i})} - \overline{D_2(x_{n+k-i}, \bar{x}_n)} \right) + \sum_{m=1}^q \lambda_m \overline{f_m(x_{n+k-i})} = \overline{h^2 D_2(x_{n+k-i}, \bar{x}_n)}, \quad i = 0(1)k$$

$$\begin{aligned}
(61) \quad & \sum_{i=0}^k \hat{\alpha}_{i,n}^F \left( D2(x_{n+k-1}, \bar{x}_{n+k-1}) - D2(x_{n+k-1}, \bar{x}_n) \right) \\
& - h^2 \sum_{i=\delta_{t0}}^{k-1} \hat{\beta}_{i,n}^F \left( D2''(x_{n+k-1}, \bar{x}_{n+k-1}) - D2''(x_{n+k-1}, \bar{x}_n) \right. \\
& \left. - D2''(x_n, \bar{x}_{n+k-1}) + D2''(x_n, \bar{x}_n) \right) + \sum_{m=1}^q \lambda_m \left( \overline{f_m''(x_{n+k-1})} - \overline{f_m''(x_n)} \right) \\
& = h^2 \left( D2''(x_n, \bar{x}_{n+k-1}) - D2''(x_n, \bar{x}_n) \right), \quad l = \delta_{t0}(1)k-1.
\end{aligned}$$

Since method (59) is interpolatory for the functions  $f_m$ ;  $m = 1(1)q$

$$(62) \quad \sum_{i=0}^k \hat{\alpha}_{i,n}^F f_m(x_{n+k-1}) - h^2 \sum_{i=\delta_{t0}}^{k-1} \hat{\beta}_{i,n}^F \left( f_m''(x_{n+k-1}) - f_m''(x_n) \right) = h^2 f_m''(x_n),$$

$$m = 1(1)q.$$

Solving the system of equations (60), (61) and (62) we can find the optimal coefficients  $\hat{\alpha}_{i,n}^F$ ,  $i = 0(1)k$  and  $\hat{\beta}_{i,n}^F$ ,  $i = \delta_{t0}(1)k-1$ .

## CHAPTER 2

### ERROR ANALYSIS OF THE OPTIMAL LINEAR MULTISTEP METHODS FOR INITIAL VALUE PROBLEM FOR SECOND ORDER DIFFERENTIAL EQUATION

#### 2.1 Introduction

This chapter is a study of certain theoretical aspects of the  $\beta$ -optimal linear multistep methods where optimization is done with respect to  $\beta$  coefficients with prefixed  $\alpha$ 's and a linear multistep method with mildly varying coefficients, implemented for an initial value problem. These aspects concern convergence analysis, magnitude of discretization error, propagation of round off error and stability analysis. The study also includes the  $\beta$  - optimal multistep methods which are interpolatory for polynomials of certain degree and other functions.

The results for the  $\beta$ -optimal methods, obtained in this chapter establish the applicability of the  $\beta$ -optimal methods to a general situation in which the solution does not necessarily belong to the underlying Hilbert space.

The space of optimization for the  $\beta$ -optimal multistep methods is  $H$ , say. The interval  $[a,b]$  of integration is a subset of the open interval  $(-r,r)$ , for the Hilbert spaces  $H^2(C_r)$  or  $L^2(\hat{C}_r)$ . The nodal points  $x_i$ 's satisfy the relation  $a = x_0 < x_1 < x_2 < \dots < x_N = b$  and they are equispaced with spacing  $h$ , with  $Nh = b-a$ . For the  $\beta$ -optimal multistep methods the optimal coefficients  $\beta_j$ 's will depend on the current variable  $x_n$  in the integration scheme. The

dependence of  $\beta_j$ 's on the point  $x_n$  is indicated by writing them as  $\beta_{j,n}$ 's.

In section 2.2, we state some results from Henrici [96], which we shall use in the subsequent sections. In section 2.3, we have given an estimate of local truncation error for the  $\beta$ -optimal multistep methods. In section 2.4, we state the relation between the optimal  $\beta$ -coefficients and the usual  $\beta$ -coefficients as  $h \rightarrow 0$ . In section 2.5, convergence of the  $\beta$ -optimal multistep method has been established. In sections 2.6 and 2.7, a bound of discretization error and a bound of propagation of round off error have been derived.

In the following sections we have also discussed about a linear multistep method with mildly varying coefficients. Using a linear multistep method with mildly varying coefficients, instead of using a linear multistep method with constant coefficients, we might not necessarily complicate the situation; rather we may gain something more. For example, for a weakly stable linear multistep method with constant coefficients, the numerical solution of the method might have uncontrolled growth. However, the growth of the numerical solution of the corresponding linear multistep method with mildly varying coefficients, could be controlled.

In section 2.8, we have established the convergence of a linear multistep method with mildly varying coefficients. In section 2.9, we have given an estimate of local truncation error for this method. In sections 2.10 and 2.11, a bound for discretization error and a bound for propagation of round off error of this method have been derived. In section 2.12, solution of

error equation of this method is derived and some criteria of stability of this method have been discussed.

We consider the following class of initial value problem for a special second order differential equation

$$(1) \quad y'' = f(x, y); \quad y(a) = \eta, \quad y'(a) = \eta',$$

on an interval  $a \leq x \leq b$ , where the function  $f(x, y)$  satisfies the conditions (i) and (ii) stated in section 1.1.

A general linear  $k$ -step method for the numerical solution of the initial value problem (1) is given by

$$(2) \quad \sum_{j=0}^k \alpha_j y_{n+j} = h^2 \sum_{j=0}^k \beta_j f_{n+j}$$

where  $y_j = y(x_j)$ ,  $f_j = f(x_j, y_j)$ ,  $x_j = a + jh$ ,  $j = 0(1)N$ ,  $\alpha_j$ ,  $\beta_j$ ,  $j = 0(1)k$ , are real constants with  $\alpha_k \neq 0$ , and  $|\alpha_j| + |\beta_j| \neq 0$ . If  $\beta_k = 0$ , the method (2) is an explicit method, and if  $\beta_k \neq 0$ , the method is an implicit one.

## 2.2 Some Well Known Results for the Usual Method

In this section, we shall state some well known results for the usual linear multistep methods for an initial value problem of the form (1), which are used in the analysis in the following sections. Let

$$(3) \quad L[y(x); h] = \sum_{i=0}^k \alpha_i y(x+ih) - h^2 \sum_{i=0}^k \beta_i y''(x+ih)$$

be the difference operator associated with the difference eqn. (2). Let  $L[y(x); h]$  be applied to functions which have continuous derivatives of sufficiently high order. Then expanding in powers of

h, we obtain

$$L[y(x);h] = C_0 y(x) + C_1 y'(x)h + \dots + C_q y^{(q)}(x)h^q + \dots,$$

where the coefficients  $C_q$  ( $q = 1, 2, \dots$ ) are independent of  $y(x)$  and are given by the following expressions.

$$C_0 = \sum_{i=0}^k \alpha_i, \quad C_1 = \sum_{i=1}^k i \alpha_i,$$

$$\text{and} \quad C_q = \frac{1}{q!} \sum_{i=1}^k i^q \alpha_i - \frac{1}{(q-2)!} \sum_{i=0}^k i^{q-2} \beta_i, \quad q=2, 3, \dots$$

Definition 1: The order  $p$  of the difference operator (3) is defined as the unique integer such that  $C_q = 0$ ,  $q = 0, 1, \dots, p+1$ ;  $C_{p+2} \neq 0$ .

Lemma 1: (Henrici [96]) Let  $L[y(x);h]$  be a difference operator of order  $p > 0$ . There exists a constant  $G > 0$ , depending only on  $L$ , such that

$$|L[y(x);h]| \leq h^{p+2} G Y, \quad a \leq x, \quad x+kh \leq b,$$

for all functions  $y(x)$  having a continuous derivatives of order  $p+2$  in  $[a, b]$ , where  $Y = \max_{a \leq x \leq b} |y^{(p+2)}(x)|$ .

Definition 2: (Henrici [96]) The linear multistep method defined by (2) is called convergent if the following statement is true for all functions  $f(x, y)$  satisfying the conditions (i) and (ii) stated in section 1.1 and all constants  $\eta$  and  $\eta'$ :

If  $y(x)$  denotes the solution of the initial value problem

$$y'' = f(x, y); \quad y(a) = \eta, \quad y'(a) = \eta'$$

then

$$\lim_{\substack{h \rightarrow 0 \\ x_n = x}} y_n = y(x)$$

holds for all  $x \in [a, b]$ , and for all sequences  $\{y_n\}$  defined by (2) with starting values

$$y_\mu = \eta_\mu(h), \quad \mu = 0, 1, 2, \dots, k-1,$$

satisfying the following two conditions

$$\lim_{h \rightarrow 0} \eta_\mu(h) = \eta, \quad \mu = 0, 1, 2, \dots, k-1,$$

$$\lim_{h \rightarrow 0} \frac{\eta_\mu(h) - \eta_0(h)}{\mu h} = \eta', \quad \mu = 0, 1, 2, \dots, k-1.$$

With the difference equation (2) or with the difference operator (3) we can associate the polynomials

$$\rho(\zeta) = \sum_{i=0}^k \alpha_i \zeta^i, \quad \text{and} \quad \sigma(\zeta) = \sum_{i=0}^k \beta_i \zeta^i.$$

Theorem 1: (Theorem 6.1, Henrici [96]) A necessary condition for the convergence of the linear multistep method (2) is that the modulus of no root of the polynomial  $\rho(\zeta)$  exceeds 1, and that the multiplicity of the roots of modulus 1 be at most 2.

The condition thus imposed on the roots of  $\rho(\zeta)$  is called the condition of stability.

Theorem 2: (Theorem 6.2, Henrici [96]) The order of a convergent linear multistep method (2) is at least 1.

This condition is called the condition of consistency.

The condition of consistency demands that the constants  $c_i$ 's



defined earlier should satisfy  $C_0 = C_1 = C_2 = 0$ , or in other words

$$\rho(1) = 0, \quad \rho'(1) = 0, \quad \rho''(1) = 2\sigma(1).$$

Lemma 2: (Lemma 6.2, Henrici [96]) Let the polynomial  $\rho(\zeta) = \alpha_k \zeta^k + \dots + \alpha_0$  satisfy the condition of stability for the integration of second order differential equation, and let the coefficients  $\gamma_l$  ( $l = 0, 1, 2, \dots$ ) be defined by

$$(4) \quad \frac{1}{\alpha_k + \alpha_{k-1}\zeta + \dots + \alpha_0 \zeta^k} = \gamma_0 + \gamma_1 \zeta + \gamma_2 \zeta^2 + \dots$$

Then there exist two constants  $\Gamma$  and  $\gamma$  such that

$$(5) \quad |\gamma_l| \leq l\Gamma + \gamma, \quad l = 0, 1, 2, \dots$$

The following lemma is for the growth of the solution of the difference equation of the form

$$(6) \quad \alpha_k z_{m+k} + \alpha_{k-1} z_{m+k-1} + \dots + \alpha_0 z_m = h^2 \{ \beta_{k,m} z_{m+k} + \beta_{k-1,m} z_{m+k-1} + \dots + \beta_{0,m} z_m \} + \lambda_m.$$

Lemma 3: (Lemma 6.3, Henrici [96]) Let the polynomial  $\rho(\zeta) = \alpha_k \zeta^k + \dots + \alpha_0$  satisfy the condition of stability for the integration of second order differential equation, and let  $B^*$ ,  $\beta$  and  $\Lambda$  be constants such that

$$(7) \quad |\beta_{k,m}| + |\beta_{k-1,m}| + \dots + |\beta_{0,m}| \leq B^*, \quad |\beta_{k,m}| \leq \beta, \quad |\lambda_m| \leq \Lambda;$$

for  $0 \leq m \leq N$ ,

and let  $0 \leq h^2 < |\alpha_k| \beta^{-1}$ . Then every solution of the difference equation (6) for which

$$(8) \quad |z_\mu| \leq Z, \quad \mu = 0, 1, 2, \dots, k-1$$

satisfies

$$(9) \quad |z_n| \leq K^* \exp \{nh^2 L^*\}, \quad 0 \leq n \leq N.$$

Here

$$(10) \quad L^* = \frac{(N\Gamma + \gamma)B^*}{1 - h^2 |\alpha_k^{-1}| \beta}, \quad \text{and} \quad K^* = \frac{(\frac{1}{2} N^2 \Gamma + N\gamma)A + (N\Gamma + \gamma)kAZ}{1 - h^2 |\alpha_k^{-1}| \beta},$$

where  $A = |\alpha_k| + |\alpha_{k-1}| + \dots + |\alpha_0|$ , and  $\Gamma$  and  $\gamma$  are defined by (5).

Theorem 3: (Theorem 6.6, Henrici [96]) The linear multistep method defined by (2) is convergent if and only if it satisfies the conditions of stability and consistency.

### 2.3 Optimal Multistep Methods for Initial Value Problems

In the following sections we shall obtain an error analysis of the optimal multistep methods when the  $\alpha$ -coefficients are kept fixed and the  $\beta$ -coefficients are optimized in a Hilbert space  $H$ , say.

Lemma 4: Let

$$L[Y(x); h] = \sum_{j=0}^k \alpha_{k-j} Y(x+(k-j)h) - h^2 \sum_{j=\delta_{t_0}}^k \beta_{k-j} Y''(x+(k-j)h)$$

be the difference operator of a  $k$ -step usual multistep method of order  $p > 0$ , and let

$$\hat{L}[Y(x); h] = \sum_{j=0}^k \alpha_{k-j} Y(x+(k-j)h) - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{k-j, x} Y''(x+(k-j)h)$$

be the difference operator of the corresponding optimal method, where  $\hat{\beta}_{k-j, x}$ ;  $0 \leq j \leq k$ , are the optimal coefficients depending on  $x$ , corresponding to the usual coefficients  $\beta_{k-j}$ ,  $0 \leq j \leq k$ . Let the

exact solution  $y(x)$  of the differential equation have a continuous derivatives of order  $p+2$  for  $x \in [a,b]$  and let

$$Y = \max_{\substack{a \leq x \leq b \\ 0 \leq i \leq p+2}} |y^{(i)}(x)|.$$

Then in a Hilbert space,  $H$ , in which  $D_x^{p+2}$ , the representer of  $p+2$ -th derivative evaluation functional at  $x$  exists and is uniformly bounded in  $[a,b]$ ,

$$|\hat{L}[y(x);h]| \leq h^{p+2} G^* Y, \quad a \leq x, \quad x+kh \leq b.$$

where  $G^*$  is a positive constant.

Proof: If  $y(x)$  has a continuous derivative of order  $p+2$  in  $[a,b]$  and if  $x$  and  $\bar{x}$  are in  $[a,b]$  then by Taylor's theorem with the integral form of the remainder,  $y(x) = P(x) + Q(x)$ , say, where

$$P(x) = y(\bar{x}) + (x-\bar{x})y'(\bar{x}) + \dots + \frac{(x-\bar{x})^{p+1}}{(p+1)!} y^{(p+1)}(\bar{x}),$$

and

$$Q(x) = \frac{1}{(p+1)!} \int_{\bar{x}}^x (x-t)^{p+1} y^{(p+2)}(t) dt.$$

Let  $\|L\|$  and  $\|\hat{L}\|$  denote the norms of the difference operators  $L$  and  $\hat{L}$ , respectively. Then since in the space of optimization,  $\|\hat{L}\| \leq \|L\|$ , we get

$$(11) \quad |\hat{L}[P(x);h]| \leq \|\hat{L}\| \|P\|$$

$$\leq \|L\| \|P\|$$

$$\leq (C_1 M_{p+2} h^{p+2}) (C_2 Y),$$

using Lemma 1, where

$$M_{p+2} = \max_{a \leq x \leq b} \| D_x^{p+2} \|,$$

$D_x^{p+2}$  being the linear operator for the  $(p+2)$ -th derivative,  $C_1$  and  $C_2$  are positive constants independent of  $x$ . Again

$$\begin{aligned} (12) \quad | \hat{L}[Q(x);h] | &\leq \| \hat{L} \| \| Q \| \\ &\leq \| L \| \| Q \| \\ &\leq (C_1 M_{p+2} h^{p+2}) (C_3 Y), \end{aligned}$$

using Lemma 1, where  $C_3$  is a positive constant. Then since,  $\hat{L}$  is a linear operator, using (11), and (12), we get

$$\begin{aligned} | \hat{L}[Y(x);h] | &= | \hat{L}[P(x)+Q(x);h] | = | \hat{L}[P(x);h] + \hat{L}[Q(x);h] | \\ &\leq | \hat{L}[P(x);h] | + | \hat{L}[Q(x);h] | \\ &\leq C_1 (C_2 + C_3) M_{p+2} h^{p+2} Y. \end{aligned}$$

So,  $| \hat{L}[Y(x);h] | \leq G^* h^{p+2} Y$ , where  $G^* = C_1 (C_2 + C_3) M_{p+2}$  is a positive constant. Hence the proof.

#### 2.4 Relation Between Optimal and Usual $\beta$ -Coefficients.

The following lemma relates the coefficients of the  $\beta$ -optimal multistep method to those of the corresponding usual multistep method.

Lemma 5 : Let

$$(13) \quad \sum_{j=0}^k \alpha_{k-j} Y_{n+k-j} = h^2 \sum_{j=\delta_{t0}}^k \beta_{k-j} f(x_{n+k-j}, Y_{n+k-j}), \quad 0 \leq n \leq N-k,$$

be a usual method of order  $p$  with constant coefficients  $\alpha_{k-j}$ 's and

$\beta_{k-j}$ 's. Let

$$(14) \quad \sum_{j=0}^k \alpha_{k-j} Y_{n+k-j} = h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{k-j,n} f(x_{n+k-j}, Y_{n+k-j}), \quad 0 \leq n \leq N-k$$

be the corresponding  $\beta$ -optimal method. Then in a Hilbert space,  $H$ , in which  $D_x^{p+2}$ , the representer of  $(p+2)$ -th derivative evaluation functional at  $x \in [a, b]$  exists and is uniformly bounded in  $[a, b]$ ,

$$|\hat{\beta}_{j,n} - \beta_j| = O(h^{p-k+\delta_{t_0}}), \quad h \rightarrow 0, \quad j = \delta_{t_0}(1)k,$$

uniformly for every  $n$ ,  $0 \leq n \leq N-k$ .

Proof: Let us consider the polynomials

$$l_{i,n}(x) = \prod_{\substack{j=\delta_{t_0} \\ j \neq i}}^k \frac{(x - x_{n+k-j})}{(x_{n+k-i} - x_{n+k-j})}, \quad i = \delta_{t_0}(1)k, \quad 0 \leq n \leq N-k,$$

of degree  $k - \delta_{t_0}$ . As  $x_{n+k-j}$ 's are in  $[a, b]$  and are equispaced,

$$\|l_{i,n}\| = O(h^{-(k-\delta_{t_0})}), \quad \text{uniformly in } i \text{ and } n.$$

Let  $M_{i,n}(x)$  be a function which satisfies the differential equation

$$M''_{i,n}(x) = l_{i,n}(x), \quad i = \delta_{t_0}(1)k; \quad 0 \leq n \leq N-k,$$

and the initial conditions  $M_{i,n}(a) = M'_{i,n}(a) = 0$ . Then

$$M'_{i,n}(x) = \int_a^x l_{i,n}(u) du = F_{i,n}(x), \quad \text{say,}$$

$$\text{and} \quad M_{i,n}(x) = \int_a^x F_{i,n}(u) du, \quad i = \delta_{t_0}(1)k; \quad 0 \leq n \leq N-k.$$

and  $M_{i,n}(x)$  is a polynomial of degree  $k+2-\delta_{t_0}$  and

$$(15) \quad \|M_{i,n}\| = O(h^{-(k-\delta_{t_0})}), \quad \text{uniformly in } i \text{ and } n.$$

Since the usual method (13) is exact for polynomials of degree

$p+1$ , it will be exact for  $M_{i,n}(x)$ ,  $i = \delta_{t_0}(1)k$ , if  $p+1 \geq k+2-\delta_{t_0}$ .

Hence

$$\sum_{j=0}^k \alpha_{k-j} M_{i,n}(x_{n+k-j}) = h^2 \sum_{j=\delta_{t_0}}^k \beta_{k-j} l_{i,n}(x_{n+k-j}).$$

Since  $l_{i,n}(x_{n+k-j}) = \delta_{ij}$ ,  $0 \leq n \leq N-k$ , we get

$$(16) \quad \sum_{j=0}^k \alpha_{k-j} M_{i,n}(x_{n+k-j}) = h^2 \beta_{k-1}, \quad i = \delta_{t_0}(1)k.$$

Let  $L[y(x);h]$  and  $\hat{L}[y(x);h]$  be the difference operators associated with the usual method (13) and the optimal method (14) respectively which represent for the local truncation error functionals of these methods. Now by applying the optimal method (14) on  $M_{i,n}(x)$ , and using (16) and the fact that  $\|\hat{L}\| \leq \|L\|$ , in  $H$ , we get

$$\left| \sum_{j=0}^k \alpha_{k-j} M_{i,n}(x_{n+k-j}) - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{k-j,n} l_{i,n}(x_{n+k-j}) \right| = |\hat{L}[M_{i,n};h]|,$$

$$\text{or,} \quad \left| \sum_{j=0}^k \alpha_{k-j} M_{i,n}(x_{n+k-j}) - h^2 \hat{\beta}_{k-1,n} \right| \leq \|\hat{L}\| \|M_{i,n}\|,$$

$$\text{or,} \quad |h^2 \beta_{k-1} - h^2 \hat{\beta}_{k-1,n}| \leq \|L\| \|M_{i,n}\|$$

$$h^2 |\beta_{k-1} - \hat{\beta}_{k-1,n}| \leq Ch^{p+2} M_{p+2} \|M_{i,n}\|,$$

where  $M_{p+2} = \max \|D_x^{p+2}\|$ ,  $D_x^{p+2}$  being the representer of  $(p+2)$ -th derivative evaluation functional. Using (15) we get

$$|\hat{\beta}_{i,n} - \beta_i| = O(h^{p-k+\delta_{t_0}}), \quad h \rightarrow 0, \quad i = \delta_{t_0}(1)k,$$

uniformly for every  $n$ ,  $0 \leq n \leq N-k$ . Hence the proof.

Remark: It is easily seen that if only  $q$  of the  $\beta_i$ 's are nonzero and are being optimized, we have instead the estimate

$$|\hat{\beta}_{i,n} - \beta_i| = O(h^{p-q+1}), \quad h \rightarrow 0, \quad i = \delta_{t_0}(1)k,$$

uniformly for every  $n$ ,  $0 \leq n \leq N-k$ .

Some examples are given below:

(i) For Cowell's method with function evaluation at three points:

$$y_{n+1} - 2y_n + y_{n-1} = h^2\{1/12 f_{n+1} + 10/12 f_n + 1/12 f_{n-1}\},$$

we have  $p = 4$ ,  $q = 3$ , so,  $|\hat{\beta}_{i,n} - \beta_i| = O(h^2)$ ,  $h \rightarrow 0$ ,  $i = 0(1)2$ .

(ii) For Stormer's method with function evaluation at one point:

$$y_{n+1} - 2y_n + y_{n-1} = h^2 f_n,$$

we have  $p = 2$ ,  $q = 1$ , so,  $|\hat{\beta}_{i,n} - \beta_i| = O(h^2)$ ,  $h \rightarrow 0$ .

(iii) For Stormer's method with function evaluation at five points:

$$y_{n+1} - 2y_n + y_{n-1} = h^2\left[\frac{299}{240}f_n - \frac{11}{15}f_{n-1} + \frac{97}{120}f_{n-2} - \frac{2}{5}f_{n-3} + \frac{19}{240}f_{n-4}\right],$$

we have  $p = 5$ ,  $q = 5$ , so,  $|\hat{\beta}_{i,n} - \beta_i| = O(h)$ ,  $h \rightarrow 0$ ,  $i = 0(1)4$ .

(iv) For a difference method of the form:

$$-y_{n-1} + y_n + y_{n+1} - y_{n+2} + 2h^2 f_{n-1} = 0,$$

we have  $p = 1$ ,  $q = 1$ , so,  $|\hat{\beta}_{0,n} - \beta_0| = O(h)$ ,  $h \rightarrow 0$ .

Assuming that  $p-k+\delta_{t_0} \geq 1$ , in the following section, we are establishing the convergence theorem using  $|\hat{\beta}_{i,n} - \beta_i| = O(h)$ .

## 2.5 Convergence of $\beta$ -Optimal Methods

Theorem 4 : In a Hilbert space  $H$ , in which  $D_x^{p+2}$ , the representer of  $(p+2)$ -th derivative evaluation functional at  $x$  exists and is

niformly bounded in  $[a, b]$ , the  $k$ -step  $\beta$ -optimal multistep method (14), corresponding to a stable and consistent usual multistep method (13), is convergent.

Proof: Let the function  $f(x, y)$  satisfy the conditions (i) and (ii) given in section 1.1 and let  $\eta$  be an arbitrary constant. Let  $y(x)$  be the solution of the initial value problem (1). Let  $y_n$ ,  $0 \leq n \leq N-k$  be the solution of the difference equation (14) defined by the starting values  $y_\mu = \eta_\mu(h)$ ,  $0 \leq \mu \leq k-1$ . We set the initial error

$$h\delta(h) = \max_{0 \leq \mu \leq k-1} | \eta_\mu(h) - y(a+\mu h) |,$$

where  $\delta(h) \rightarrow 0$  as  $h \rightarrow 0$ .

Let  $L[y(x); h]$  and  $\hat{L}[y(x); h]$  be the difference operators associated with the usual method (13) and the optimal method (14) respectively, for the point  $x = x_m$ ,  $0 \leq m \leq N-k$ . Let

$$\| y'' \| = \max_{x \in [a, b]} | y''(x) |.$$

Using Lemma 5, we get for the point  $x = x_m$ ,

$$\begin{aligned} |\hat{L}[y(x_m); h] - L[y(x_m); h]| &= \left| h^2 \sum_{j=\delta_{t0}}^k (\hat{\beta}_{k-j, m} - \beta_{k-j}) y''(x_{m+k-j}) \right| \\ &\leq h^2 \| y'' \| \sum_{j=\delta_{t0}}^k |\hat{\beta}_{k-j, m} - \beta_{k-j}| = o(h^3). \end{aligned}$$

Since  $L$  is consistent,

$$| L[y(x_m); h] | = o(h^2).$$

$$\text{So, } | \hat{L}[y(x_m); h] | \leq | L[y(x_m); h] | + o(h^3) = o(h^2),$$

$$\text{or, } | \hat{L}[y(x_m); h] | = \theta_m h^2 \theta(h),$$



where  $|\theta_m| \leq 1$ , and  $\theta(h) \rightarrow 0$  as  $h \rightarrow 0$ . Thus we get for  $0 \leq m \leq N-k$ ,

$$(17) \quad \sum_{j=0}^k \alpha_{k-j} Y(x_{m+k-j}) - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{k-j,m} f(x_{m+k-j}, Y(x_{m+k-j})) = \theta_m h^2 \theta(h).$$

The numerical solution satisfies the following relation at  $x_m$

$$(18) \quad \sum_{j=0}^k \alpha_{k-j} Y_{m+k-j} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{k-j,m} f(x_{m+k-j}, Y_{m+k-j}) = 0.$$

Let

$$(19) \quad e_m = Y_m - Y(x_m), \quad 0 \leq m \leq N-k, \quad \text{and}$$

$$(20) \quad g_m = \begin{cases} [f(x_m, Y_m) - f(x_m, Y(x_m))] e_m^{-1}, & \text{if } e_m \neq 0, \\ 0, & \text{if } e_m = 0. \end{cases}$$

Subtracting (18) from (17), we get the error equation

$$(21) \quad \sum_{j=0}^k \alpha_{k-j} e_{m+k-j} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{k-j,m} g_{m+k-j} e_{m+k-j} = \theta_m h^2 \theta(h).$$

By Lipschitz condition (ii) of  $f$ , given in section 1.1,

$|g_m| \leq L$ ,  $0 \leq m \leq N-k$ . Now we can apply Lemma 3, with

$$z_m = e_m, \quad Z = h \delta(h), \quad \Lambda = h^2 \theta(h), \quad N = (x_m - a) h^{-1}, \quad A = \sum_{j=0}^k |\alpha_{k-j}|,$$

$$\beta_{k-j,m} = \hat{\beta}_{k-j,m} g_{m+k-j}, \quad j = \delta_{t0}(1)K, \quad \beta = \sup_{m,h} |\hat{\beta}_{k,m}|,$$

$$\beta = \sup_{m,h} \sum_{j=\delta_{t0}}^k |\hat{\beta}_{k-j,m}| < \infty, \quad \sum_{j=\delta_{t0}}^k |\beta_{k-j,m}| = \sum_{j=\delta_{t0}}^k |\hat{\beta}_{k-j,m}| |g_{m+k-j}| \leq BL.$$

For a later use, let us also put

$$(22) \quad B^* = BL.$$

Then  $|\beta_{k,m}| = |\hat{\beta}_{k,m} g_{m+k}| \leq L |\hat{\beta}_{k,m}| \leq \beta L$ . Let

$$(23) \quad \Gamma^* = \Gamma [1 - h^2 L |\alpha_k^{-1}| \beta]^{-1}, \quad a^* = a - h\gamma \Gamma^{-1},$$

then  $\frac{1}{2} N^2 \Gamma + N\gamma \leq \frac{1}{2} (x_n - a^*)^2 \Gamma h^{-2}$ , and  $N\Gamma + \gamma = (x_n - a^*) \Gamma h^{-1}$ .

Thus for  $h^2 < |\alpha_k| L^{-1} \beta^{-1}$ , we get

$$|e_n| \leq \Gamma^* \left\{ (x_n - a^*) k A \delta(h) + \frac{1}{2} (x_n - a^*)^2 \theta(h) \right\} \exp \left\{ (x_n - a^*)^2 \Gamma^* B^* \right\}.$$

Since  $\delta(h) \rightarrow 0$  and  $\theta(h) \rightarrow 0$  as  $h \rightarrow 0$ ,  $e_n \rightarrow 0$  as  $h \rightarrow 0$ . This completes the proof.

## 2.6 A Bound for the Discretization Error

Let us assume that the exact solution  $y(x)$  has a continuous derivative of order  $p+2$  for  $x \in [a, b]$ . Set

$$(24) \quad Y = \max_{a \leq x \leq b} |y^{(p+2)}(x)|.$$

Instead of the difference equation (14), let the numerical solution  $y_n$  satisfy the following equation, incorporating round offs:

$$(25) \quad \sum_{j=0}^k \alpha_{k-j} y_{m+k-j} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{k-j,m} f(x_{m+k-j}, y_{m+k-j}) = \theta_m K h^{q+2}, \quad 0 \leq m \leq N-k,$$

where  $|\theta_m| \leq 1$ , and  $K, q$  are constants with  $q > 0$ . The term involving  $K$  stands for small error in the implementation of the difference scheme. Let us set the initial errors

$$(26) \quad |y_\mu - y(x_\mu)| \leq h\delta(h), \quad \mu = 0(1)k-1.$$

Theorem 5 : Under the conditions (24), (25) and (26), if  $h^2 < |\alpha_k| L^{-1} \beta^{-1}$ , the discretization error,  $e_n = y_n - y(x_n)$ , satisfies

$$|e_n| \leq \Gamma^* \left\{ (x_n - a^*) k A \delta(h) + \frac{1}{2} (x_n - a^*)^2 \{ K h^q + G^* y h^p \} \right\} \exp \left\{ (x_n - a^*)^2 \Gamma^* B^* \right\},$$

for  $a \leq x_n \leq b$ , where  $\Gamma^*$ ,  $a^*$  are defined in (23),  $B^*$  is defined in (22),  $p$  is the order of the difference operator  $L[y(x); h]$  associated with the usual method (13),  $G^*$  is a constant depending only on  $\hat{L}[y(x); h]$ , the difference operator associated with the optimal method (14).

Proof: The exact solution  $y(x)$  satisfies the following relation at the point  $x_m$

$$(27) \quad \sum_{j=0}^k \alpha_{k-j} y(x_{m+k-j}) - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{k-j,m} f(x_{m+k-j}, y(x_{m+k-j})) = \hat{T}_m, \quad 0 \leq m \leq N-k,$$

where  $\hat{T}_m$  is the local truncation error at the point  $x_m$  for the optimal method (14). By Lemma 4,

$$|\hat{T}_m| \leq G^* y h^{p+2}.$$

Hence,  $\hat{T}_m = \theta_{m1} G^* y h^{p+2}$ , where  $|\theta_{m1}| \leq 1$ . Let  $e_m$  and  $g_m$  be defined by (19) and (20). Subtracting (27) from (25), we get the error equation

$$(28) \quad \sum_{j=0}^k \alpha_{k-j} e_{m+k-j} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{k-j,m} g_{m+k-j} e_{m+k-j} = \theta'_m (K h^{q+2} + G^* y h^{p+2}),$$

where  $|\theta'_m| \leq 1$ . Now applying Lemma 3 to the difference equation (28), proceed as in the proof of Theorem 4, with the only exception that  $\Lambda = K h^{q+2} + G^* y h^{p+2}$ . Then for  $h^2 < |\alpha_k| L^{-1} \beta^{-1}$ , we get the following relation:

$$|e_n| \leq \Gamma^* \left\{ (x_n - a^*) k A \delta(h) + \frac{1}{2} (x_n - a^*)^2 \{ K h^q + G^* y h^p \} \right\} \exp \left\{ (x_n - a^*)^2 \Gamma^* B^* \right\}.$$

Remark: If  $\delta(h) = O(h^p)$  and  $q \geq p$ , then  $e_n = O(h^p)$ .

## 2.7 An A-Priori Bound for Round-Off Error

Let  $\tilde{y}_n$  denote the numerical solution in the presence of round-offs corresponding to the exact numerical solution  $y_n$ . Let  $\varepsilon_n$  denote the round-off in the difference equation related with  $\tilde{y}_n$ .

Theorem 6: If  $h^2 < |\alpha_k| L^{-1} \beta^{-1}$ , the round-off error  $r_n = \tilde{y}_n - y_n$  satisfies for  $a \leq x_n \leq b$ ,

$$|r_n| \leq \varepsilon h^{-2} \Gamma^* \frac{1}{2} (x_n - a^*)^2 \exp \left\{ (x_n - a^*)^2 \Gamma^* B^* \right\}, \quad 0 \leq n \leq N-k,$$

where  $\Gamma^*$  and  $a^*$  are defined in (23),  $B^*$  is defined in (22),  $L$  is Lipchitz constant for the function  $f(x, y)$  and  $\varepsilon$  is a bound for local round-off errors.

Proof: The numerical approximation  $\tilde{y}_m$  to  $y_m$ , the exact solution of the difference equation (14), satisfies the following equation

$$(29) \quad \sum_{j=0}^k \alpha_{k-j} \tilde{y}_{m+k-j} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{k-j,m} f(x_{m+k-j}, \tilde{y}_{m+k-j}) = \varepsilon_{m+k}, \quad 0 \leq m \leq N-k,$$

where  $\varepsilon_{m+k}$  is the local round-off, which depends on the computational procedure and on the organization of the arithmetic unit of the computing equipment. Let  $|\varepsilon_m| \leq \varepsilon$ , where  $\varepsilon$  is independent of  $m$ .

Let  $r_m = \tilde{y}_m - y_m$  be the accumulated round-off error at the point  $x_m$ , and let

$$g_m = \begin{cases} r_m^{-1} [f(x_m, \tilde{y}_m) - f(x_m, y_m)], & \text{if } r_m \neq 0, \\ 0, & \text{if } r_m = 0. \end{cases}$$

The numerical solution  $y_m$  at  $x_m$  satisfies the following relation

$$(30) \quad \sum_{j=0}^k \alpha_{k-j} y_{m+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{k-j,m} f(x_{m+k-j}, y_{m+k-j}) = 0, \quad 0 \leq m \leq N-k.$$

Subtracting (30) from (29), we get an error equation for  $r_m$ ,

$$(31) \quad \sum_{j=0}^k \alpha_{k-j} r_{m+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{k-j,m} g_{m+k-j} r_{m+k-j} = \varepsilon_{m+k}, \quad 0 \leq m \leq N-k.$$

Now we can apply Lemma 3 to the relation (31) with

$$z_m = r_m, \quad \Lambda = \varepsilon, \quad N = (x_m - a)h^{-1}, \quad A = \sum_{j=0}^k |\alpha_j|,$$

$Z = 0$ , (i.e., there is no initial round-off error),

$$\beta_{k-j,m} = \hat{\beta}_{k-j,m} g_{m+k-j}, \quad j = \delta_{t_0}(1)k,$$

$$B = \sup_{m,h} \sum_{j=\delta_{t_0}}^k |\hat{\beta}_{k-j,m}|; \quad \sum_{j=\delta_{t_0}}^k |\beta_{k-j,m}| \leq BL = B^*, \text{ say.}$$

$$\text{Let } \beta = \sup_{m,h} |\hat{\beta}_{k,m}|; \quad \text{then } |\beta_{k,m}| = |\hat{\beta}_{k,m} g_{m+k}| \leq \beta L.$$

$\Gamma$  and  $\gamma$  are as defined in Lemma 2. Then proceeding as in the proof of Theorem 4, assuming that  $h^2 < |\alpha_k| L^{-1} \beta^{-1}$ ,

$$|r_n| \leq \varepsilon h^{-2} \Gamma^* \frac{1}{2} (x_n - a^*)^2 \exp \left( (x_n - a^*)^2 \Gamma^* B^* \right).$$

Hence the proof.

## 2.8 Linear Multistep Method with Mildly Varying Coefficients

In this section we shall study a linear multistep method with mildly varying coefficients. The basic motivation comes from the work of Lambert [121] on multistep methods with mildly varying coefficients for first order initial value problems and the fact that our optimal multistep methods also fall in this category. We can interpret this situation as perturbations in  $\alpha$  and  $\beta$ -coefficients of the usual method (2). Such a method for the initial value problem (1) in a neighbourhood of the initial point  $a$ , is given by

$$(32) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_n) y_{n+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_n) f_{n+j},$$

with  $\tilde{\alpha}_j(x) = \alpha_j + h^2 a_j(x)$ , and  $\tilde{\beta}_j(x) = \beta_j + h b_j(x)$ , where  $\alpha_j, \beta_j$  are constants and  $|a_j(x)| \leq A_1$ , and  $|b_j(x)| \leq B_1$ , for all  $x \in [a, b]$ ,  $0 \leq j \leq k$ , and  $A_1, B_1$  are finite constants.

Let us define the linear difference operators  $L$  and  $M$  by

$$L[y(x); h] = \sum_{j=0}^k \alpha_j y(x+jh) - h^2 \sum_{j=0}^k \beta_j y''(x+jh),$$

and

$$M[y(x); x; h] = \sum_{j=0}^k a_j(x) y(x+jh) - h \sum_{j=0}^k b_j(x) y''(x+jh).$$

Then the linear difference operator associated with the perturbed method (32) is given by

$$\tilde{L}[y(x); x; h] = \sum_{j=0}^k \{\alpha_j + h^2 a_j(x)\} y(x+jh) - h^2 \sum_{j=0}^k \{\beta_j + h b_j(x)\} y''(x+jh)$$

$$\begin{aligned}
&= \sum_{j=0}^k \alpha_j Y(x+jh) - h^2 \sum_{j=0}^k \beta_j Y''(x+jh) + \\
&\quad + h^2 \left( \sum_{j=0}^k a_j(x) Y(x+jh) - h \sum_{j=0}^k b_j(x) Y''(x+jh) \right) \\
&= L[Y(x);h] + h^2 M[Y(x);x;h].
\end{aligned}$$

$L$  is the operator associated with the linear multistep method (2) with constant coefficients. Its order  $p$  is defined in Definition 1.

If  $M$  is applied to a sufficiently differentiable function  $y(x)$ , then by Taylor's series expansion we have

$$\begin{aligned}
M[Y(x);x;h] &= \sum_{j=0}^k a_j(x) Y(x+jh) - h \sum_{j=0}^k b_j(x) Y''(x+jh) \\
&= D_0(x) + hD_1(x) + h^2 D_2(x) + \dots + h^r D_r(x) + \dots,
\end{aligned}$$

where

$$D_0(x) = a_0(x) + a_1(x) + a_2(x) + \dots + a_k(x),$$

$$D_1(x) = \{a_1(x) + 2a_2(x) + 3a_3(x) + \dots + ka_k(x)\} Y'(x)$$

$$- \{b_0(x) + b_1(x) + b_2(x) + \dots + b_k(x)\} Y''(x), \text{ and}$$

$$D_q(x) = \left( \frac{1}{q!} a_1(x) + \frac{2^q}{q!} a_2(x) + \dots + \frac{k^q}{q!} a_k(x) \right) Y^{(q)}(x)$$

$$- \left( \frac{1}{(q-1)!} b_1(x) + \frac{2^{q-1}}{(q-1)!} b_2(x) + \dots + \frac{k^{q-1}}{(q-1)!} b_k(x) \right) Y^{(q+1)}(x),$$

for  $q = 2, 3, \dots$

Since  $\tilde{L}[Y(x);x;h] = L[Y(x);h] + h^2 M[Y(x);x;h]$ , by Taylor's series expansion we get

$$\tilde{L}[Y(x);x;h] = c_0 Y(x) + h c_1 Y'(x) + h^2 c_2 Y''(x) + h^3 c_3 Y^{(3)}(x) + \dots$$

$$+h^2\left\{D_0(x) + hD_1(x) + h^2D_2(x) + \dots\right\},$$

so that

$$\begin{aligned} \frac{1}{h^2}\tilde{L}[y(x);x;h] &= \frac{1}{h^2}\left\{C_0y(x)+hC_1y'(x)+h^2C_2y''(x)+h^3C_3y^{(3)}(x)+\dots\right\} \\ &\quad + \left\{D_0(x)+hD_1(x)+h^2D_2(x)+\dots\right\}. \end{aligned}$$

Since the left hand side is a difference approximation for the given differential equation, the right hand side should vanish as  $h \rightarrow 0$ . Thus we get the following definition.

Definition 3: The perturbed method (32) is consistent if  $C_0 = C_1 = C_2 = 0$ , and  $D_0(x)=0$ , for all  $x \in [a,b]$ .

Definition 4: The perturbed method (32) associated with the operator  $\tilde{L}$  is said to be stable if the operator  $L$  is stable.

Theorem 7: A sufficient condition for the method (32) to be convergent is that it be stable and consistent.

Proof : Let the function  $f$  satisfy the conditions (i) and (ii) stated in section 1.1 and let  $\eta$  and  $\eta'$  be arbitrary constants. Let  $y(x)$  be the solution of the initial value problem

$$y'' = f(x,y), \quad y(a) = \eta, \quad y'(a) = \eta'.$$

Let  $y_n$ , ( $n=0,1,2,\dots$ ) be the solution of the difference scheme (32) defined by the starting values

$$y_\mu = \eta_\mu(h), \quad \mu = 0,1,2,\dots,k-1.$$

We set

$$\delta_1 = h\delta(h) = h\delta = \max_{0 \leq \mu \leq k-1} |\eta_\mu(h) - y(a+\mu h)|,$$



$$\delta_2 = \max_{0 \leq \mu \leq k-1} \left| \frac{\eta_\mu(h) - \eta_0(h)}{\mu h} - y(a + \mu h) \right|,$$

and assume that  $\lim_{h \rightarrow 0} \delta(h) = 0$ , and  $\lim_{h \rightarrow 0} \delta_2 = 0$ . Then we have to show

that for any  $x$  in  $[a, b]$

$$\lim_{\substack{h \rightarrow 0 \\ x_n = x}} y_n = y(x).$$

The function  $y''(x) = f(x, y)$  is continuous in  $[a, b]$ . Let us define the function

$$\chi(\varepsilon) = \max_{\substack{x^* \\ |x^* - x| < \varepsilon \\ x^*, x \in [a, b]}} |y''(x^*) - y''(x)|.$$

We know,

$$(33) \quad y''(x_{m+\mu}) = y''(x_m) + \theta_\mu(\mu h), \quad \text{for } \mu = 0, 1, \dots, k,$$

where  $|\theta_\mu| \leq 1$ . Again, since

$y(x_{m+\mu}) = y(x_m) + \mu h y'(x_m) + \frac{(\mu h)^2}{2!} y''(\xi_\mu)$ , where  $x_m < \xi_\mu < x_{m+\mu}$ , we get

$$(34) \quad y(x_{m+\mu}) = y(x_m) + \mu h y'(x_m) + \frac{(\mu h)^2}{2!} \left( y''(x_m) + \theta'_\mu \chi(\mu h) \right),$$

where  $|\theta'_\mu| \leq 1$ . Using (33) and (34), we get

$$\begin{aligned} L[y(x_m); h] &= \sum_{j=0}^k \alpha_j y(x_m + jh) - h^2 \sum_{j=0}^k \beta_j y''(x_m + jh) \\ &= (\alpha_0 + \alpha_1 + \dots + \alpha_k) y(x_m) + (\alpha_1 + 2\alpha_2 + \dots + k\alpha_k) h y'(x_m) \\ &\quad + \left\{ \left( \frac{1}{2!} \alpha_1 + \frac{2^2}{2!} \alpha_2 + \dots + \frac{k^2}{2!} \alpha_k \right) - (\beta_0 + \beta_1 + \dots + \beta_k) \right\} h^2 y''(x_m) \\ &\quad + \theta' \left\{ \frac{1}{2!} |\alpha_1| + \frac{2^2}{2!} |\alpha_2| + \dots + \frac{k^2}{2!} |\alpha_k| \right\} h^2 \chi(kh) \end{aligned}$$

$$- \theta \left\{ |\beta_0| + |\beta_1| + \dots + |\beta_k| \right\} h^2 \chi(kh),$$

where  $|\theta'| \leq 1$  and  $|\theta| \leq 1$ . By consistency of  $L$  we have,

$$|L[y(x_m); h]| \leq Kh^2 \chi(kh),$$

where  $K = (\frac{1}{2!}|\alpha_1| + \frac{2^2}{2!}|\alpha_2| + \dots + \frac{k^2}{2!}|\alpha_k|) + (|\beta_0| + |\beta_1| + \dots + |\beta_k|)$ .

Now if  $M$  is applied to a sufficiently differentiable function  $y(x)$ , then

$$\begin{aligned} M[y(x_m); x_m; h] &= \sum_{j=0}^k a_j(x_m) y(x_m + jh) - h^2 \sum_{j=0}^k b_j(x_m) y''(x_m + jh) \\ &= (a_0(x_m) + a_1(x_m) + \dots + a_k(x_m)) y(x_m) \\ &\quad + (a_1(x_m) + 2a_2(x_m) + \dots + ka_k(x_m)) hy'(x_m) \\ &\quad + \left\{ \frac{1}{2!}a_1(x_m) + \frac{2^2}{2!}a_2(x_m) + \dots + \frac{k^2}{2!}a_k(x_m) \right\} h^2 y''(x_m) \\ &\quad + \phi' \left\{ \frac{1}{2!}|a_1(x_m)| + \frac{2^2}{2!}|a_2(x_m)| + \dots + \frac{k^2}{2!}|a_k(x_m)| \right\} h^2 \chi(kh) \\ &\quad - (b_0(x_m) + b_1(x_m) + \dots + b_k(x_m)) hy''(x_m) \\ &\quad - \phi \left\{ |b_0(x_m)| + |b_1(x_m)| + \dots + |b_k(x_m)| \right\} h \chi(kh), \end{aligned}$$

where  $|\phi'| \leq 1$  and  $|\phi| \leq 1$ .

For all  $x \in [a, b]$ , we have the following :

(i)  $|a_j(x_n)| \leq A_1$ , and  $|b_j(x_n)| \leq B_1$ ,  $0 \leq j \leq k$ ;

(ii) by consistency,  $D_0(x_m) = a_0(x_m) + a_1(x_m) + \dots + a_k(x_m) = 0$ ;

(iii)  $|y''(x)| \leq F_2$ , where  $F_2$  is a finite constant.

Since  $y''(x) = f(x, y(x))$  is a continuous function of  $x$  on the closed

interval  $[a, b]$ , it is necessarily bounded on  $[a, b]$ , and

$$|Y'(x)| = |\eta' + \int_a^x Y''(x) dx| \leq |\eta'| + \int_a^x |Y''(x)| dx \leq |\eta'| + F_2(b-a) = F_1, \quad \text{say.}$$

Using (i), (ii) and (iii) we have

$$\begin{aligned} |M[Y(x_m); x_m; h]| &\leq A_1 h F_1 \frac{k(k+1)}{2} + A_1 h^2 F_2 \frac{k(k+1)(2k+1)}{12} \\ &\quad + A_1 h^2 \chi(kh) \frac{k(k+1)(2k+1)}{12} + B_1 h F_2 \frac{k(k+1)}{2} + B_1 h \chi(kh) \frac{k(k+1)}{2} \\ &= Gh, \quad \text{say,} \end{aligned}$$

$$\begin{aligned} G &= A_1 F_1 \frac{k(k+1)}{2} + A_1 h F_2 \frac{k(k+1)(2k+1)}{12} + A_1 h \chi(kh) \frac{k(k+1)(2k+1)}{12} \\ &\quad + B_1 F_2 \frac{k(k+1)}{2} + B_1 \chi(kh) \frac{k(k+1)}{2} = O(1). \end{aligned}$$

Then

$$\begin{aligned} |\tilde{L}[Y(x_m); x_m; h]| &\leq |L[Y(x_m); h]| + h^2 |M[Y(x_m); x_m; h]| \\ &\leq Kh^2 \chi(kh) + Gh^3. \end{aligned}$$

So, we can write

$$\tilde{L}[Y(x_m); x_m; h] = \theta_m \left( Kh^2 \chi(kh) + Gh^3 \right), \quad \text{where } |\theta_m| \leq 1, \quad \text{or,}$$

$$(33) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_m) Y(x_{m+j}) - h^2 \sum_{j=0}^k \tilde{\beta}_j(x_m) Y''(x_{m+j}) = \theta_m \left( Kh^2 \chi(kh) + Gh^3 \right).$$

Let us define

(34)  $e_m = y_m - Y(x_m)$ , the discretization error at the point  $x_m$ ; and

$$(35) \quad g_m = \begin{cases} [f(x_m, y_m) - f(x_m, Y(x_m))] e_m^{-1}, & \text{if } e_m \neq 0, \\ 0, & \text{if } e_m = 0. \end{cases}$$

Subtracting (35) from the corresponding relation:

$$\sum_{j=0}^k \tilde{\alpha}_j(x_m) y_{m+j} - h^2 \sum_{j=0}^k \tilde{\beta}_j(x_m) f_{m+j} = 0,$$

we get the following difference equation for  $e_m$ :

$$\sum_{j=0}^k \tilde{\alpha}_j(x_m) e_{m+j} - h^2 \sum_{j=0}^k \tilde{\beta}_j(x_m) g_{m+j} e_{m+j} = \theta_m \left( Kh^2 \chi(kh) + Gh^3 \right).$$

Let us rewrite this equation as

$$(38) \quad \sum_{j=0}^k \alpha_j e_{m+j} - h^2 \sum_{j=0}^k [\tilde{\beta}_j(x_m) g_{m+j} - a_j(x_m)] e_{m+j} = \theta_m \left( Kh^2 \chi(kh) + Gh^3 \right).$$

By Lipchitz condition,  $|g_{m+j}| \leq L$ ,  $j = 0(1)k$ ,  $m = 0, 1, 2, \dots$ . Now we can apply Lemma 3 to (38) with

$$z_m = e_m, \quad Z = h\delta(h), \quad \Lambda = Kh^2 \chi(kh) + Gh^3, \quad N = (x_m - a)h^{-1}, \quad A = \sum_{j=0}^k |\alpha_j|,$$

$$B = \sum_{j=0}^k |\beta_j|, \quad \beta_{j,m} = \tilde{\beta}_j(x_m) g_{m+j} - a_j(x_m), \quad j = 0(1)k,$$

$$\sum_{j=0}^k |\beta_{j,m}| \leq \sum_{j=0}^k \left\{ \left( |\beta_j| + h|b_j(x_m)| \right) |g_{m+j}| + |a_j(x_m)| \right\}$$

$$\leq (B + h(k+1)B_1)L + (k+1)A_1, \quad \text{and let}$$

$$(39) \quad B^* = [B + h(k+1)B_1]L + (k+1)A_1$$

$$|\beta_{k,m}| \leq (|\beta_k| + h|b_k(x_m)|) |g_{m+k}| + |a_k(x_m)| \leq (|\beta_k| + hB_1)L + A_1 = \beta, \quad \text{say,}$$

and with  $\Gamma$  and  $\gamma$  as defined in Lemma 2. Let

$$(40) \quad \Gamma^* = \frac{\Gamma}{1 - h^2 |\alpha_k^{-1}| \beta}, \quad \text{and} \quad a^* = a - \frac{h\gamma}{\Gamma}.$$

Then  $N\Gamma + \gamma = (x_n - a^*)\Gamma h^{-1}$ , and  $\frac{1}{2}N^2\Gamma + N\gamma \leq \frac{1}{2}(x_n - a^*)^2\Gamma h^{-2}$ , and we

have

$$|e_n| \leq \Gamma^* \left( (x_n - a^*) k A \delta + \frac{1}{2} (x_n - a^*)^2 [K\chi(kh) + Gh] \right) \exp \left( (x_n - a^*)^2 \Gamma^* B^* \right).$$

Since,  $\delta \rightarrow 0$  as  $h \rightarrow 0$ , and  $\chi(kh) \rightarrow 0$  as  $h \rightarrow 0$ , we have  $e_n \rightarrow 0$  as  $h \rightarrow 0$ ,  $n \rightarrow \infty$ . Hence the convergence.

## 2.9 A Bound for the Local Truncation Error

Lemma : Let

$$L[Y(x); h] = \sum_{j=0}^k \alpha_j Y(x+jh) - h^2 \sum_{j=0}^k \beta_j y''(x+jh)$$

be the difference operator of order  $p > 0$ , corresponding to the usual method (2) and let

$$\tilde{L}[Y(x); x; h] = \sum_{j=0}^k \tilde{\alpha}_j(x) Y(x+jh) - h^2 \sum_{j=0}^k \tilde{\beta}_j(x) y''(x+jh)$$

be the difference operator corresponding to the perturbed method (32). Let  $y(x)$  possess a continuous derivative of order  $\tilde{p}+2$  on the interval  $[a, b]$ , where  $\tilde{p} = \min\{p, r+1\}$ ,  $r$  is an integer for which the forthcoming system of equations (41-i)-(41-ii) and (42-i)-(42-ii) hold, and set

$$\tilde{Y} = \max_{\substack{x \in [a, b] \\ 0 \leq i \leq \tilde{p}+2}} |y^{(i)}(x)|.$$

Then

$$|\tilde{L}[Y(x); x; h]| \leq h^{\tilde{p}+2} \tilde{G} \tilde{Y},$$

uniformly in  $x$ ,  $x+kh \in [a, b]$ , where  $\tilde{G}$  is a constant independent of  $y$ .

Proof: If  $y(x)$  has continuous derivatives of order  $n+2$  in  $[a, b]$

and if  $x$  and  $\bar{x}$  are in  $[a, b]$ , then by Taylor's theorem with the integral form of the remainder, we have

$$y(\bar{x}) = y(x) + (\bar{x}-x)y'(x) + \frac{(\bar{x}-x)^2}{2!}y''(x) + \dots + \frac{(\bar{x}-x)^{n+1}}{(n+1)!}y^{(n+1)}(x) \\ + \frac{1}{(n+1)!} \int_x^{\bar{x}} (\bar{x}-t)^{n+1}y^{(n+2)}(t)dt.$$

Now

$$\tilde{L}[y(x); x; h] = L[y(x); h] + h^2 M[y(x); x; h].$$

Since  $L[y(x); h]$  is of order  $p > 0$ , by Lemma 1, we have

$$|L[y(x); h]| \leq h^{p+2} G Y_1,$$

uniformly in  $x$ ,  $x+kh \in [a, b]$  where  $G$  is a constant independent of  $y$  and

$$Y_1 = \max_{x \in [a, b]} |y^{(p+2)}(x)|.$$

Now if  $M[y(x); x; h]$  is applied to a function  $y(x)$ , having continuous derivative of order  $r+1$ , then

$$M[y(x); x; h] = \sum_{j=0}^k a_j(x)y(x+jh) - h^2 \sum_{j=0}^k b_j(x)y''(x+jh) \\ = (a_0(x) + a_1(x) + \dots + a_k(x))y(x) + \\ + h \left[ \{a_1(x) + 2a_2(x) + \dots + ka_k(x)\}y'(x) - \{b_0(x) + b_1(x) + \dots + b_k(x)\}y^{(2)}(x) \right] \\ + h^2 \left[ \left\{ \frac{1}{2!}a_1(x) + \frac{2^2}{2!}a_2(x) + \dots + \frac{k^2}{2!}a_k(x) \right\}y^{(2)}(x) \right. \\ \left. - \{b_1(x) + 2b_2(x) + \dots + kb_k(x)\}y^{(3)}(x) \right] + \dots \\ + h^r \left[ \left\{ \frac{1}{r!}a_1(x) + \frac{2^r}{r!}a_2(x) + \dots + \frac{k^r}{r!}a_k(x) \right\}y^{(r)}(x) \right]$$

$$\begin{aligned}
& - \left\{ \frac{1}{(r-1)!} b_1(x) + \frac{2^{r-1}}{(r-1)!} b_2(x) + \dots + \frac{k^{r-1}}{(r-1)!} b_k(x) \right\} y^{(r+1)}(x) \Big] \\
& + h^{r+1} \left[ \left\{ \frac{1}{(r+1)!} a_1(x) + \frac{2^{r+1}}{(r+1)!} a_2(x) + \dots + \frac{k^{r+1}}{(r+1)!} a_k(x) \right\} y^{(r+1)}(x) \right. \\
& + \frac{1}{(r+1)!} \left[ a_1(x) \int_x^{x+h} (x+h-t)^{r+1} y^{(r+2)}(t) dt + a_2(x) \int_x^{x+2h} (x+2h-t)^{r+1} y^{(r+2)}(t) dt \right. \\
& \quad \left. + \dots + a_k(x) \int_x^{x+kh} (x+kh-t)^{r+1} y^{(r+2)}(t) dt \right] \\
& - \frac{h^2}{(r-1)!} \left[ b_1(x) \int_x^{x+h} (x+h-t)^{r-1} y^{(r+2)}(t) dt + b_2(x) \int_x^{x+2h} (x+2h-t)^{r-1} y^{(r+2)}(t) dt \right. \\
& \quad \left. + \dots + b_k(x) \int_x^{x+kh} (x+kh-t)^{r-1} y^{(r+2)}(t) dt \right].
\end{aligned}$$

Let  $a_i(x)$ ,  $b_i(x)$ ;  $i = 0(1)k$  satisfy the following equations:

$$(41-i) \quad a_0(x) + a_1(x) + \dots + a_k(x) = 0,$$

$$(41-ii) \quad \frac{1}{t!} a_1(x) + \frac{2^t}{t!} a_2(x) + \dots + \frac{k^t}{t!} a_k(x) = 0, \quad t = 1(1)r+1, \text{ and}$$

$$(42-i) \quad b_0(x) + b_1(x) + \dots + b_k(x) = 0,$$

$$(42-ii) \quad \frac{1}{t!} b_1(x) + \frac{2^t}{t!} b_2(x) + \dots + \frac{k^t}{t!} b_k(x) = 0, \quad t = 1(1)r-1.$$

Then, putting  $t = x+sh$ , we have

$$\begin{aligned}
M[y(x); x; h] &= \frac{h^{r+2}}{(r+1)!} \sum_{i=0}^k a_i(x) \int_0^1 (i-s)^{r+1} y^{(r+2)}(x+sh) ds \\
&\quad - \frac{h^{r+1}}{(r-1)!} \sum_{i=0}^k b_i(x) \int_0^1 (i-s)^{r-1} y^{(r+2)}(x+sh) ds.
\end{aligned}$$

$$\text{Hence,} \quad |M[y(x); x; h]| \leq G_1 h^{r+1} Y_1,$$

where  $G_1$  is independent of  $y$ , and  $Y_1 = \max_{x \in [a, b]} |y^{(r+2)}(x)|$ .

Hence,  $|\tilde{L}[y(x);x;h]| \leq |L[y(x);h]| + h^2 |M[y(x);x;h]|$

$$\leq h^{p+2}GY + G_1 h^{r+3}Y_1 \leq h^{\tilde{p}+2}\tilde{G}\tilde{Y} = O(h^{\tilde{p}+2}),$$

where  $\tilde{p} = \min\{p, r+1\}$ ,  $\tilde{Y} = \max\{Y, Y_1\}$  and  $\tilde{G} = \max\{G, G_1\}$ .

## 2.10 A Bound for the Discretization Error

Let us assume that the exact solution  $y(x)$  has a continuous derivative of order  $\tilde{p}+2$  for  $x \in [a, b]$ . Set

$$(43) \quad \tilde{Y} = \max_{a \leq x \leq b} |y^{(\tilde{p}+2)}(x)|,$$

where  $\tilde{p} = \min(p, r+1)$ ;  $p (> 0)$  is the order of the usual operator  $L[y(x);h]$  and  $r$  is a positive integer for which the systems of equations (41-i)-(41-ii) and (42-i)-(42-ii) hold good.

Instead of the difference equation (32) let the numerical solution  $y_n$  satisfy the following equation, at a point  $x_m$

$$(44) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_m) y_{m+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_m) f(x_{m+j}, y_{m+j}) + \theta_m K h^{q+2}, \quad 0 \leq m \leq N-k,$$

where  $|\theta_m| \leq 1$ , and  $K, q$  are constants with  $q > 0$ . The term involving  $K$  stands for small error in the difference equation. Let us set the starting errors

$$(45) \quad |y_\mu - y(x_\mu)| \leq h\delta(h), \quad \mu = 0(1)k-1$$

Theorem 8 : Under the conditions (43), (44) and (45) if  $h^2 < |\alpha_k \beta^{-1}|$ , the discretization errors  $e_n = y_n - y(x_n)$  satisfy the following relation, for  $a \leq x_n \leq b$ ,



$$|e_n| \leq \Gamma^* \left\{ (x_n - a^*) k A \delta(h) + \frac{1}{2} (x_n - a^*)^2 \{ K h^q + \tilde{G} \tilde{Y} h^{\tilde{p}} \} \exp \left\{ (x_n - a^*)^2 \Gamma^* B^* \right\} \right\},$$

where  $\Gamma^*$ ,  $a^*$  are defined in (40),  $B^*$  is defined in (39).  $G^*$  is a constant depending only on  $\tilde{L} [y(x); h]$ , the difference operator associated with the optimal method (32) and  $L$  is the Lipchitz constant.

Proof: The exact solution  $y(x)$  satisfies the following relation at the point  $x_m$

$$(46) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_m) y(x_m + jh) = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_m) f(x_m + jh, y(x_m + jh)) + \tilde{T}_m, \quad 0 \leq m \leq N-k,$$

where  $\tilde{T}_m$  is the local truncation error functional for the optimal method (32) at the point  $x_m$ . By Lemma 6, we have

$$|\tilde{T}_m| \leq \tilde{G} \tilde{Y} h^{\tilde{p}+2}.$$

Hence,  $\tilde{T}_m = \theta_{m1} \tilde{G} \tilde{Y} h^{\tilde{p}+2}$ , where  $|\theta_{m1}| \leq 1$ .

Let  $e_m$  and  $g_m$  be defined by (36) and (37). Subtracting (46) from (44), we get the error equation

$$(47) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_m) e_{m+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_m) g_{m+j} e_{m+j} + \theta'_m (K h^{q+2} + \tilde{G} \tilde{Y} h^{\tilde{p}+2}),$$

where  $|\theta'_m| \leq 1$ . Now applying Lemma 3 to the difference equation (47), and proceeding as in Theorem 7, with the only exception that  $\Lambda = K h^{q+2} + \tilde{G} \tilde{Y} h^{\tilde{p}+2}$ , we get for  $h^2 < |\alpha_k \beta^{-1}|$ ,

$$|e_n| \leq \Gamma^* \left\{ (x_n - a^*) k A \delta(h) + \frac{1}{2} (x_n - a^*)^2 \{ K h^q + \tilde{G} \tilde{Y} h^{\tilde{p}} \} \exp \left\{ (x_n - a^*)^2 \Gamma^* B^* \right\} \right\}.$$

Remark: If  $\delta(h) = O(h^{\tilde{p}})$  and  $q \geq \tilde{p}$ , then  $e_n = O(h^{\tilde{p}})$ .

## 2.11 Propagation of Round-Off Error

Theorem 9: If  $h^2 < |\alpha_k \beta^{-1}|$ , the round-off error  $r_n = \tilde{y}_n - y_n$  satisfies the following relation, for  $a \leq x_n \leq b$ ,

$$|r_n| \leq \varepsilon h^{-2} \Gamma^* \frac{1}{2} (x_n - a^*)^2 \exp\left((x_n - a^*)^2 \Gamma^* B^*\right), \quad 0 \leq n \leq N-k,$$

where  $\Gamma^*$  and  $a^*$  are defined in (40),  $B^*$  is defined in (39),  $L$  is the Lipchitz constant for the function  $f(x, y)$  and  $\varepsilon$  is a bound for the local round-off errors.

Proof: The numerical approximations  $\{\tilde{y}_n\}$  to  $\{y_n\}$  of the numerical method (32) satisfy the following equations,

$$(48) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_n) \tilde{y}_{n+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_n) f(x_{n+j}, \tilde{y}_{n+j}) + \varepsilon_{n+k}, \quad 0 \leq n \leq N-k,$$

where  $\varepsilon_n$  is the local round-off error which depends on the computational procedure and on the organization of the arithmetic unit of the computing equipment.

Let  $r_n = \tilde{y}_n - y_n$  be the accumulated round-off error at a point  $x_n$ . Let  $|\varepsilon_n| \leq \varepsilon$ , where  $\varepsilon$  is independent of  $n$ . Let

$$r_m = \begin{cases} r_m^{-1} [f(x_m, \tilde{y}_m) - f(x_m, y_m)], & \text{if } r_m \neq 0, \\ 0, & \text{if } r_m = 0. \end{cases}$$

Subtracting (46) from the corresponding relation:

$$\sum_{j=0}^k \tilde{\alpha}_j(x_n) y_{n+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_n) f(x_{n+j}, y_{n+j}),$$

we get the following equation:

$$\sum_{j=0}^k \tilde{\alpha}_j(x_n) r_{n+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_n) g_{n+j} r_{n+j} + \varepsilon_{n+k},$$

Rewriting this equation in the form:

$$\sum_{j=0}^k \alpha_j r_{n+j} = h^2 \sum_{j=0}^k \{\tilde{\beta}_j(x_n) g_{n+j} - a_j(x_n)\} r_{n+j} + \varepsilon_{n+k},$$

and applying Lemma 3 to this equation, with  $z_m = r_m$ ,  $Z = 0$  (i.e. there is no initial round-off error),  $\Lambda = \varepsilon$ ,  $N = (x_m - a)h^{-1}$ ,

$$\beta_{jm} = \tilde{\beta}_j(x_m) g_{m+j} - a_j(x_m) = \{\beta_j + h b_j(x_m)\} g_{m+j} - a_j(x_m), \quad j = 0(1)k,$$

$$A = \sum_{j=0}^k |\alpha_j|, \quad B = \sum_{j=0}^k |\beta_j|, \quad \text{we get, as in the proof of Theorem 7,}$$

$$\sum_{j=0}^k |\beta_{jm}| \leq (B + h(k+1)B_1)L + (k+1)A_1 = B^*, \quad \text{say,}$$

$$|\beta_{km}| \leq (|\beta_k| + hB_1)L + A_1 = \beta, \quad \text{say, and } \Gamma^* = \frac{\Gamma}{1 - h^2 L |\alpha_k^{-1}| \beta}, \quad a^* = a - \frac{h\gamma}{\Gamma}.$$

Thus assuming that  $h^2 < |\alpha_k| \beta^{-1}$ , we have

$$|r_n| \leq \varepsilon h^{-2} \Gamma^* \frac{1}{2} (x_n - a^*)^2 \exp\left((x_n - a^*)^2 \Gamma^* B^*\right). \quad \text{Hence the proof.}$$

## 2.12 Stability Analysis of Linear Multistep Methods With Varying Coefficients

Let the equation (32) be a linear multistep method with mild varying coefficients corresponding to a consistent and stable linear multistep method (2) with constant coefficients.

Let at a point  $x_n$  the exact solution  $y(x)$  of the differential equation satisfy

$$(49) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_n) y(x_{n+j}) = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_n) f(x_{n+j}, y(x_{n+j})) + T_n,$$

where  $T_n$  is the local truncation error at  $x_n$ . If  $\{y_n\}$  denotes the computed solution of the difference equation (49), then

$$(50) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_n) y_{n+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_n) f_{n+j} + R_n,$$

where  $R_n$  is the local round off error at  $x_n$ . Subtracting (50) from (49), and applying mean value theorem we get the following equation for the discretisation error  $e_n = y(x_n) - y_n$ :

$$(51) \quad \sum_{j=0}^k \tilde{\alpha}_j(x_n) e_{n+j} = h^2 \sum_{j=0}^k \tilde{\beta}_j(x_n) \frac{\partial f}{\partial y}(x_{n+j}, \xi_{n+j}) e_{n+j} + E_n,$$

where  $\xi_{n+j} \in (y_{n+j}, y(x_{n+j}))$  and  $E_n = T_n - R_n$ .

Let us assume that  $\frac{\partial f}{\partial y}(x_{n+j}, \xi_{n+j}) = -q(x_{n+j})$ ,  $j = 0(1)k$ , i.e., the function  $f$  is linear in  $y$ . Then the error equation (51) becomes,

$$\sum_{j=0}^k \left\{ \tilde{\alpha}_j(x_n) + h^2 q(x_{n+j}) \tilde{\beta}_j(x_n) \right\} e_{n+j} = E_n,$$

$$\text{or,} \quad \sum_{j=0}^k \left\{ \alpha_j + h^2 \left[ a_j(x_n) + q(x_{n+j}) [\beta_j + h b_j(x_n)] \right] \right\} e_{n+j} = E_n,$$

or,

$$(52) \quad \sum_{j=0}^k \alpha_j e_{n+j} = E_n - h^2 \sum_{j=0}^k \phi_j(n) e_{n+j},$$

where  $\phi_j(n) = a_j(x_n) + q(x_{n+j}) [\beta_j + h b_j(x_n)]$ ,  $j = 0(1)k$ . Without loss of generality, we can assume that  $\alpha_k = 1$ .

Equation (52) is a linear difference equation with constant coefficients with an inhomogeneous term represented by the whole right hand side. The complete solution of such an equation may be expressed as the sum of the complementary function and a particular

solution; moreover, the latter may be represented as the convolution of the inhomogeneous term with the particular solution of the reduced equation (53), given below.

Let  $\{\eta_n\}$  be the complete solution of the reduced equation,

$$(53) \quad \sum_{j=0}^k \alpha_j e_{n+j} = 0.$$

The characteristic equation associated with the difference equation (53) is given by

$$\rho(\zeta) = \sum_{j=0}^k \alpha_j \zeta^j = 0.$$

Assume that the usual linear multistep method (2) is consistent and stable. By its consistency, we have

$$\rho(1) = 0, \rho'(1) = 0, \rho''(1) = 2\sigma(1) \neq 0,$$

i.e.,  $\zeta = 1$  is a double root of the equation  $\rho(\zeta) = 0$ . Let this root be  $\zeta = \zeta_1$ . If the other roots of  $\rho(\zeta) = 0$  are distinct, say  $\zeta_\mu$ ,  $3 \leq \mu \leq k$ , then

$$\eta_n = (A_1 + nA_2)\zeta_1^n + \sum_{p=3}^k A_p \zeta_p^n = (A_1 + nA_2) + \sum_{p=3}^k A_p \zeta_p^n.$$

In general, if the distinct roots, other than 1, are  $\zeta_\mu$ ,  $3 \leq \mu \leq \tilde{k}$ , say, let the root  $\zeta_\mu$ , of  $\rho(\zeta) = 0$ , have the multiplicity  $p_\mu$ ,  $3 \leq \mu \leq \tilde{k}$ , so that  $\sum_{\mu} p_\mu = k-2$ . Then

$$(54) \quad \eta_n = (A_1 + nA_2) + \sum_{3 \leq \mu \leq \tilde{k}} \left\{ B_{\mu,0} + nB_{\mu,1} + n(n-1)B_{\mu,2} + \dots + n(n-1) \dots (n-p_\mu+2)B_{\mu,p_\mu-1} \right\} \zeta_\mu^n$$

where the coefficients:

$$A_1, A_2, \left\{ B_{\mu, i}, i = 0(1)p_{\mu}-1, 3 \leq \mu \leq \tilde{k} \right\}$$

are to be determined through the initial values  $e_0, e_1, e_2, \dots, e_{k-1}$ ; where the particular solution  $\{z_n\}$  of equation (53) satisfies the initial conditions  $z_0 = z_1 = \dots = z_{k-2} = 0, z_{k-1} = 1$ , we have

$$(55) \quad e_n = \eta_n + \sum_{t=1}^n z_{n-t} d_t = \eta_n + \sum_{t=1}^{n-1} z_{n-t} d_t,$$

where

$$(56) \quad d_t = E_{t-1} - h^2 \sum_{j=0}^k \phi_j(t-1) e_{t+j-1}, \quad t \geq 0.$$

Following Lambert [121], let us define

$$(57) \quad d_t^* = E_{t-1} - h^2 \sum_{j=0}^k \phi_j(t-1) \eta_{t+j-1}.$$

Then by (56) and (57), we have

$$\begin{aligned} d_t^* - d_t &= h^2 \sum_{j=0}^k \phi_j(t-1) \left\{ e_{t+j-1} - \eta_{t+j-1} \right\} \\ &= h^2 \sum_{j=0}^k \phi_j(t-1) \left\{ \eta_{t+j-1} + \sum_{s=1}^{t+j-2} z_{t+j-1-s} d_s - \eta_{t+j-1} \right\} \\ &= h^2 \sum_{j=0}^k \phi_j(t-1) \left\{ \sum_{s=1}^{t+j-2} z_{t+j-1-s} d_s \right\} \\ &= h^2 \sum_{s=1}^{t+k-2} \left\{ \sum_{j=0}^k \phi_j(t-1) z_{t+j-1-s} \right\} d_s. \end{aligned}$$

We take  $z_n$  to be zero, when  $n$  is a negative integer. From the definition of  $z_n$ , we see that the expression in braces in the last expression vanishes when  $t < s$ , and hence we may write for  $k \geq 2$ ,

$$(58) \quad d_t^* = d_t + h^2 \sum_{s=1}^t \Phi_{ts} d_s, \quad t = 1, 2, \dots, T,$$

where  $\Phi_{ts} = \sum_{j=0}^k \phi_j(t-1) z_{t+j-1-s}$  and  $T \geq 1$ . For any fixed  $T \leq N$ , let us define the vectors

$$d = (d_1, d_2, \dots, d_T)^T, \quad \text{and} \quad d^* = (d_1^*, d_2^*, \dots, d_T^*)^T.$$

Since  $\Phi_{ts} = 0$  when  $s > t$ ,  $\Phi$  is a lower triangular matrix. Then

$$d^* = d + h^2 \Phi d = (I + h^2 \Phi) d.$$

Since  $z_n$  is an  $\eta_n$ , where the constants in  $\eta_n$  are to be determined by the conditions  $z_0 = z_1 = \dots = z_{k-2} = 0$ ,  $z_{k-1} = 1$ , and since the usual method is stable,  $z_n = O(n)$ . Again from (53),  $\phi_j(t-1) = O(1)$ .

So,  $\Phi_{ts} = O(t)$ . But  $\Phi_{tt} = \sum_{j=0}^k \phi_j(t-1) z_{j-1} = O(1)$ .

Hence for all sufficiently small  $h$ ,  $(I + h^2 \Phi)^{-1}$  exists and

$$d = (I + h^2 \Phi)^{-1} d^*,$$

or,

$$(59) \quad d_t = \sum_{s=1}^t \psi_{ts} d_s^*, \quad t = 1, 2, \dots, T.$$

$\Phi$  being a lower triangular matrix,  $\Psi = (I + h^2 \Phi)^{-1} = (\psi_{ts})$  is also so. Hence  $\psi_{ts} = 0$ , when  $s > t$ ,  $t = 1, 2, \dots, T$ . So, the solution (55) can be written as

$$(60) \quad e_n = \eta_n + \sum_{t=1}^{n-1} z_{n-t} \left\{ \sum_{s=1}^t \psi_{ts} d_s^* \right\}.$$

Now from (57) and (54), we have

$$d_t^* = E_{t-1} - h^2 \sum_{j=0}^k \phi_j(t-1) \left[ A_1 + (t+j-1)A_2 + \sum_{\mu} \left\{ B_{\mu,0} + (t+j-1)B_{\mu,1} + \right. \right. \\ \left. \left. (t+j-1)(t+j-2)B_{\mu,2} + \dots + (t+j-1)(t+j-2) \dots (t+j+1-p_{\mu})B_{\mu,p_{\mu}-1} \right\} \zeta_{\mu}^{t+j-1} \right],$$

$$\text{or, } d_t^* = E_{t-1} - h^2 \left[ A_1 F_1(t-1) + A_2 F_2(t-1) + \sum_{\mu} \zeta_{\mu}^{t-1} F_{\mu}(t-1) \right],$$

where

$$(61-i) \quad F_1(t-1) = \sum_{j=0}^k \phi_j(t-1), \quad F_2(t-1) = \sum_{j=0}^k (t+j-1) \phi_j(t-1),$$

and for  $3 \leq \mu \leq \tilde{k}$ ,

$$\begin{aligned} (61-ii) \quad F_{\mu}(t-1) &= B_{\mu,0} \sum_{j=0}^k \phi_j(t-1) \zeta_{\mu}^j + B_{\mu,1} \sum_{j=0}^k (t+j-1) \phi_j(t-1) \zeta_{\mu}^j + \\ &+ B_{\mu,2} \sum_{j=0}^k (t+j-1)(t+j-2) \phi_j(t-1) \zeta_{\mu}^j + \dots + \\ &+ B_{\mu,p_{\mu}-1} \sum_{j=0}^k (t+j-1)(t+j-2) \dots (t+j-p_{\mu}+1) \phi_j(t-1) \zeta_{\mu}^j \\ &= B_{\mu,0} F_{\mu,0}(t-1) + B_{\mu,1} F_{\mu,1}(t-1) + \dots + B_{\mu,p_{\mu}-1} F_{\mu,p_{\mu}-1}(t-1), \end{aligned}$$

$$\text{where } F_{\mu,0}(t-1) = \sum_{j=0}^k \phi_j(t-1) \zeta_{\mu}^j, \quad \text{and}$$

$$F_{\mu,i}(t-1) = \sum_{j=0}^k (t+j-1)(t+j-2) \dots (t+j-i) \phi_j(t-1) \zeta_{\mu}^j, \quad i=1(1)p_{\mu}-1.$$

Then

$$e_n = (A_1 + nA_2)$$

$$+ \sum_{\mu} \left\{ B_{\mu,0} + nB_{\mu,1} + n(n-1)B_{\mu,2} + \dots + n(n-1) \dots (n-p_{\mu}+2)B_{\mu,p_{\mu}-1} \right\} \zeta_{\mu}^n$$

$$+ \sum_{t=1}^{n-1} z_{n-t} \left[ \sum_{s=1}^t \psi_{ts} \left\{ E_{s-1} - h^2 \left( A_1 F_1(s-1) + A_2 F_2(s-1) + \sum_{\mu} \zeta_{\mu}^{s-1} F_{\mu}(s-1) \right) \right\} \right],$$

or,

$$(62) \quad e_n = A_1 \left( 1 - h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} F_1(s-1) \right) + A_2 \left( n - h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} F_2(s-1) \right)$$



$$\begin{aligned}
& + \sum_{\mu} \left[ B_{\mu,0} \left\{ \zeta_{\mu}^n - h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} \zeta_{\mu}^{s-1} F_{\mu,0}(s-1) \right\} \right. \\
& + B_{\mu,1} \left\{ n \zeta_{\mu}^n - h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} \zeta_{\mu}^{s-1} F_{\mu,1}(s-1) \right\} + \dots \\
& + B_{\mu,p_{\mu}-1} \left\{ n(n-1) \dots (n-p_{\mu}+2) \zeta_{\mu}^n - h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} \zeta_{\mu}^{s-1} F_{\mu,p_{\mu}-1}(s-1) \right\} \Big] \\
& + \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} E_{s-1},
\end{aligned}$$

or,

$$\begin{aligned}
(63) \quad e_n = & A_1 [1 + P_1(n)] + A_2 [n + P_2(n)] + \sum_{\mu} \left[ B_{\mu,0} \left\{ \zeta_{\mu}^n + P_{\mu,0}(n) \right\} + \right. \\
& B_{\mu,1} \left\{ n \zeta_{\mu}^n + P_{\mu,1}(n) \right\} + \dots + B_{\mu,p_{\mu}-1} \left\{ n(n-1) \dots (n-p_{\mu}+2) \zeta_{\mu}^n + P_{\mu,p_{\mu}-1}(n) \right\} \\
& \left. + \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} E_{s-1} \right],
\end{aligned}$$

where

$$(64) \quad \begin{cases} P_j(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} F_j(s-1), \quad j = 1, 2 \\ P_{\mu,i}(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} \zeta_{\mu}^{s-1} F_{\mu,i}(s-1), \quad i=0(1)p_{\mu}-1, \quad 3 \leq \mu \leq \tilde{k}. \end{cases}$$

Comparing (62) and (63), we see that

$$P_1(n), P_2(n), \left\{ P_{\mu,0}(n), P_{\mu,1}(n), \dots, P_{\mu,p_{\mu}-1}(n); 3 \leq \mu \leq \tilde{k} \right\}$$

may be interpreted as the perturbation corresponding to the fundamental solutions:

$$\zeta_1^n (= 1), n\zeta_1^n (= n), \left\{ \zeta_\mu^n, n\zeta_\mu^n, \dots, n(n-1)(n-p_\mu+2)\zeta_\mu^n, 3 \leq \mu \leq \tilde{k} \right\}$$

of the equation

$$\sum_{j=0}^k \alpha_j e_{n+j} = 0.$$

Now we shall find the order of magnitude of  $\psi_{ts}$ . For this,  $\psi_{tt} = (1+h^2\Phi_{tt})^{-1} = O(1)$ , since  $\Phi_{tt} = O(1)$ . Assume that  $\psi_{ts} = O(h^2)$ , for  $s < t$ ,  $t = 2, 3, \dots, q$ ; for convenience of notation, identifying  $\Phi_{t,s}$  with  $\Phi_{ts}$  and  $\psi_{t,s}$  with  $\psi_{ts}$ , by (58) and (59), we get

$$d_{q+1}^* = d_{q+1} + h^2 \sum_{s=1}^{q+1} \Phi_{q+1,s} d_s,$$

$$\text{or, } d_{q+1}^* = (1 + h^2 \Phi_{q+1,q+1}) d_{q+1} + h^2 \sum_{s=1}^q \Phi_{q+1,s} \left( \sum_{u=1}^s \psi_{su} d_u^* \right),$$

$$\text{or, } d_{q+1}^* = (1 + h^2 \Phi_{q+1,q+1}) d_{q+1} + h^2 \sum_{u=1}^q \left\{ \sum_{s=u}^q \Phi_{q+1,s} \psi_{su} \right\} d_u^*,$$

or,

$$d_{q+1} = \frac{1}{(1+h^2\Phi_{q+1,q+1})} d_{q+1}^* - \frac{h^2}{(1+h^2\Phi_{q+1,q+1})} \sum_{u=1}^q \left\{ \sum_{s=u}^q \Phi_{q+1,s} \psi_{su} \right\} d_u^*.$$

Again from (59), we get

$$d_{q+1} = \sum_{s=1}^{q+1} \psi_{q+1,s} d_s^* = \psi_{q+1,q+1} d_{q+1}^* + \sum_{s=1}^q \psi_{q+1,s} d_s^*.$$

Comparing these two relations for  $d_{q+1}$ , we get

$$(65) \quad \begin{cases} \psi_{q+1,q+1} = \frac{1}{(1+h^2\Phi_{q+1,q+1})}, \\ \psi_{q+1,u} = \frac{-h^2}{(1+h^2\Phi_{q+1,q+1})} \sum_{s=u}^q \Phi_{q+1,s} \psi_{su}, \quad u = 1(1)q. \end{cases}$$

Now, since  $q \leq n$ , and  $\Phi_{q+1,s} = O(n)$ , and by hypothesis,  $\psi_{su} = O(h^2)$ , for  $u < s \leq q$

$$\psi_{q+1,u} = O(h^2), \quad u = 1(1)q.$$

To complete the induction, we note that indeed

$$\psi_{21} = -h^2 \Phi_{21} (1 + h^2 \Phi_{11})^{-1} (1 + h^2 \Phi_{22})^{-1} = O(h^2).$$

Hence we conclude that

$$\psi_{ts} = O(h^2), \text{ as } h \rightarrow 0, \text{ for } s < t.$$

Since by the stability of the usual method (2), we know

$$|\zeta_\mu| \leq 1, \quad 1 \leq \mu \leq k,$$

and  $\psi_{ts} = O(h^2)$ , for  $s < t$ , and  $t \leq n$ , from (61-i) and (61-ii), we observe that:

if  $F_j(s-1) = O(h)$ ,  $j=1,2$  and  $F_{\mu,i}(s-1) = O(h)$ ,  $i = 0(1)p_\mu - 1$ , for all  $s$

then  $\sum_{s=1}^t \psi_{ts} F_j(s-1) = O(h)$ , for  $j = 1,2$ , and

$$\sum_{s=1}^t \psi_{ts} \zeta_\mu^{s-1} F_{\mu,i}(s-1) = O(h), \quad i = 0(1)p_\mu - 1, \text{ for all } s.$$

Since  $z_n = O(n)$ , and  $t \leq n$ , from (64), we have

$$P_j(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \left( \sum_{s=1}^t \psi_{ts} F_j(s-1) \right) = O(h), \quad \text{for } j = 1,2;$$

and

$$P_{\mu,i}(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \left( \sum_{s=1}^t \psi_{ts} \zeta_\mu^{s-1} F_{\mu,i}(s-1) \right) = O(h), \quad i = 0(1)p_\mu - 1,$$

$$3 \leq \mu \leq \tilde{k}.$$

Now we shall study some stability criteria of the multistep method (32) with mildly varying coefficients. For solving an

initial value problem (1), according to a usual method, the computed numerical solution satisfies at a point  $x_n$ :

$$(67) \quad \sum_{j=0}^k \alpha_j y_{n+j} = h^2 \sum_{j=0}^k \beta_j f_{n+j} + R_n,$$

where  $R_n$  is the round off error at  $x_n$ . The true solution satisfies

$$(68) \quad \sum_{j=0}^k \alpha_j y(x_{n+j}) = h^2 \sum_{j=0}^k \beta_j f(x_{n+j}, y(x_{n+j})) + T_n,$$

where  $T_n$  is the local truncation error at  $x_n$ . Subtracting (67) from (68), and applying Mean Value Theorem, we get the equation for the discretisation error  $e_n = y(x_n) - y_n$ ,

$$\sum_{j=0}^k \alpha_j e_{n+j} = h^2 \sum_{j=0}^k \beta_j \frac{\partial f}{\partial y}(x_{n+j}, \xi_{n+j}) e_{n+j} + E_n,$$

where  $\xi_{n+j} \in (y_{n+j}, y(x_{n+j}))$  and  $E_n = T_n - R_n$ . Assume that  $E_n = E$ , a constant and  $\frac{\partial f}{\partial y}(x, y) = -q$ , a constant, then the above error equation becomes

$$(69) \quad \sum_{j=0}^k (\alpha_j + h^2 q \beta_j) e_{n+j} = E.$$

Let  $\zeta_{\mu h}$ ,  $\mu = 1(1)k$  be the  $k$  distinct roots of the polynomial

$$\sum_{j=0}^k (\alpha_j + h^2 q \beta_j) \zeta^j = 0,$$

or, 
$$\rho(\zeta) + h^2 q \sigma(\zeta) = 0,$$

where 
$$\rho(\zeta) = \sum_{j=0}^k \alpha_j \zeta^j \quad \text{and} \quad \sigma(\zeta) = \sum_{j=0}^k \beta_j \zeta^j;$$

then the solution of the error equation (69) is

$$(70) \quad e_n = \sum_{\mu=1}^k A_{\mu} \zeta_{\mu h}^n + (\text{particular solution involving } E).$$

Let  $\zeta_\mu$ ,  $\mu = 1(1)k$ , be the  $k$  roots of the equations  $\rho(\zeta) = 0$ . Then by the stability of the usual method (2),  $|\zeta_\mu| \leq 1$ ,  $\mu = 1(1)k$ . By its consistency,  $\rho(1) = 0$ ,  $\rho'(1) = 0$ , So,  $\zeta = 1$  is a double root of the equation  $\rho(\zeta) = 0$ .

Definition 3: We define a multistep method of the form (2) to be weakly stable if there is no other double root of the equation  $\rho(\zeta) = 0$ , on the unit circle, except  $\zeta = 1$ .

Now for a weakly stable usual method (2),  $\zeta_{\mu h}$ , the root of the equation  $\rho(\zeta) + h^2\sigma(\zeta) = 0$ , corresponding to a simple root  $\zeta_\mu$  of the equation  $\rho(\zeta) = 0$  on the unit circle, may lie outside the unit circle, and its contribution  $\zeta_{\mu h}^n$  to the error given by (70) will produce an uncontrolled growth.

Now let us consider the situation if we use the perturbed difference method (32) instead of the usual method (2) for the numerical solution of the initial value problem (1). Suppose the usual method (2) is weakly stable, and let  $\zeta_\mu$  be a simple root of the equation  $\rho(\zeta) = 0$  on the unit circle. In this situation if we can choose  $a_j(x)$  and  $b_j(x)$  in such a way that

$$F_{\mu,0}(s-1) = O(h) \text{ for all } s, \quad 3 \leq \mu \leq \tilde{k},$$

then the perturbation  $P_{\mu,0}(n)$  of the corresponding fundamental solution  $\zeta_\mu^n$  of the equation :

$$\sum_{j=0}^k \alpha_j e_{n+j} = 0,$$

will remain under control and will be of  $O(h)$ .

Let us consider other stability criteria of the method (32).

The solution of an initial value problem of the form

$$y'' = \lambda y, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0,$$

is given by

$$y(x) = C_1 \exp\{\sqrt{\lambda}x\} + C_2 \exp\{-\sqrt{\lambda}x\},$$

$$\text{where } C_1 = \frac{(y_0 \sqrt{\lambda} + y'_0) \exp\{-\sqrt{\lambda}t_0\}}{2\sqrt{\lambda}}, \quad C_2 = \frac{(y_0 - y'_0) \exp\{\sqrt{\lambda}t_0\}}{2\sqrt{\lambda}}.$$

If the initial conditions  $y(t_0)$  and  $y'(t_0)$  are so chosen that  $C_2 = 0$ , e.g., if  $y_0 = y'_0$ , then  $y(x) = C_1 \exp\{\sqrt{\lambda}x\}$ , and  $y(x) \rightarrow 0$ , for all  $x$ , for the points of the set  $\{\lambda: \operatorname{Re}\sqrt{\lambda} < 0\}$ .

Now we define  $A_1$  stability as follows.

Definition 4 : A linear multistep method of the form (2) applied to an initial value problem of the form

$$(71) \quad y'' = \lambda y, \quad y(t_0) = y'(t_0) = y_0,$$

where  $\lambda$  is a complex constant such that  $\operatorname{Re}\{\sqrt{\lambda}\} < 0$ , is called  $A_1$  stable if all the solutions of this method tend to zero, as  $h \rightarrow 0$ ,  $n \rightarrow \infty$ , for the critical region,  $x_n = o(h^{-1/2})$ .

(Note that asking  $y_n \rightarrow 0$ , for general  $x_n \rightarrow \infty$ , is too restrictive a condition.)

Now let us apply a perturbed linear multistep method of the form (32), where  $\frac{\partial f}{\partial y} = -q(x_n) = \lambda$ , a complex constant, to an initial value of the form (71) with  $\operatorname{Re}\{\sqrt{\lambda}\} < 0$ , then the true solution of the initial value problem (71),  $y(x) \rightarrow 0$ , for all  $x \rightarrow \infty$ .

The discretisation error,  $e_n = y_n - y(x_n)$ , at the point  $x_n$  is given by the expression (64) or (65) where  $\frac{\partial f}{\partial y} = -q(x_n)$  is being replaced by  $\lambda$  in the expressions for  $\phi_j(s-1)$ , that is,

$$\phi_j(s-1) = a_j(x_{s-1}) - h\lambda\{\beta_j + hb_j(x_{s-1})\}.$$

Then  $F_j(s-1)$ ,  $j = 1, 2$  and  $F_{\mu,1}(s-1)$ ,  $i = 0(1)p_{\mu}-1$ , given by (61-i) and (61-ii) are with this definition of  $\phi_j(s-1)$ .

The  $k$  constants

$$A_1, A_2, \left\{ B_{\mu,1}, i = 0(1)p_{\mu}-1, 3 \leq \mu \leq \tilde{k} \right\}$$

of the error solution (63) are to be determined through the initial values of errors  $e_0, e_1, \dots, e_{k-1}$ , using some one-step method. If

$$(72) \quad e_i = y_i - y(x_i) = O(h^2), \quad i = 0(1)k-1,$$

that is, the initial values  $y_i$ ,  $i = 0(1)k-1$ , are accurate of  $O(h^2)$  to the actual solution  $y(x_i)$ ,  $i = 0(1)k-1$ , then the above constants will have the order of magnitude of  $O(h^2)$ .

By stability of the usual method (2), we have  $|\zeta_{\mu}| \leq 1$ . Since  $\psi_{ts} = O(h^2)$ , for  $s < t$ , and  $\psi_{tt} = O(1)$ ,  $t \leq n$ , we observe that if

$$(73-i) \quad F_j(s-1) = O(1), \quad j = 1, 2, \quad \text{and}$$

$$(73-ii) \quad F_{\mu,1}(s-1) = O(1), \quad i = 0(1)p_{\mu}-1, \quad \text{for all } s,$$

$$\text{then} \quad \sum_{s=1}^t \psi_{ts} F_j(s-1) = O(1), \quad \text{for } j = 1, 2,$$

$$\text{and} \quad \sum_{s=1}^t \psi_{ts} \zeta_{\mu}^{s-1} F_{\mu,1}(s-1) = O(1), \quad i = 0(1)p_{\mu}-1, \quad \text{for all } s.$$

Since  $z_n = O(n)$  and  $t \leq n$ , assuming that  $nh = o(h^{-1/2})$ ,

$$P_j(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} F_j(s-1) = o(h^{-1}), \quad \text{for } j = 1, 2, \quad \text{and}$$

$$P_{\mu,1}(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} \zeta_{\mu}^{s-1} F_{\mu,1}(s-1) = o(h^{-1}), \quad i = 0(1)p_{\mu}-1,$$

$$\text{for } 3 \leq \mu \leq \tilde{k}.$$

We know that due to the local truncation error for the method at  $x_n$  is

$$T_n = O(h^3), \quad \text{if } r \geq 2, \quad \text{where } r \text{ is defined as earlier.}$$

If the round-off error  $R_n$  of the method satisfies the following relation:

$$(74) \quad R_n = O(h^3),$$

$$\text{then} \quad E_n = T_n - R_n = O(h^3).$$

Since  $\psi_{ts} = O(h^2)$ , for  $s < t$ , and  $t \leq n$ , and  $z_n = O(n)$ , and  $\psi_{tt} = O(1)$ , and  $nh = o(h^{-1/2})$ , we have

$$\sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} E_{s-1} = o(1).$$

Then from (63), we see that under the assumptions (72), (73-i), (73-ii) and (74)  $e_n \rightarrow 0$  as  $h \rightarrow 0$ , in the critical region  $x_n = o(h^{-1/2})$ .

Let us assume that the given usual method (2) is  $A_1$ -stable. Then under the assumptions (72), (73-i), (73-ii) and (74), the numerical solution  $y_n = y(x_n) + e_n$  of the perturbed linear multistep method (32) applied to a model initial value problem (71) with  $\text{Re}\{\sqrt{\lambda}\} < 0$ , tends to zero, as  $h \rightarrow 0$ ,  $n \rightarrow \infty$  and  $x_n = o(h^{-1/2})$ , provided  $r \geq 2$ . Hence the perturbed linear multistep method (32) is  $A_1$ -stable under the assumptions (72), (73-i), (73-ii) and (74) in the critical region  $x_n = o(h^{-1/2})$ , provided  $r \geq 2$ .



Now we are defining  $A_2$  stability as follows.

Definition 5 : A linear multistep method (2) when applied to an initial value problem of the form (71), where  $\lambda$  is a complex constant, is called  $A_2$ -stable if all the solutions of this method are bounded as  $h \rightarrow 0$ ,  $n \rightarrow \infty$ , in the larger critical region  $x_n = O(h^{-1/2})$ , provided  $r \geq 2$ .

Suppose, the perturbed linear multistep method (32) is applied to a model initial value problem of the form (71), then the solution of its error equation (53), is given by (63), where  $\frac{\partial f}{\partial y} = -q(x_n)$ , is being replaced by  $\lambda$  in the expressions for  $F_1(s-1)$ ,  $F_2(s-1)$  and  $F_{\mu,1}(s-1)$ ,  $i = O(1)p_\mu - 1$ ,  $3 \leq \mu \leq \tilde{k}$ , given by (61-i) and (61-ii). If

$$(75) \quad e_i = y_i - Y(x_i) = O(h^2), \quad i = O(1)k-1,$$

then the constants  $A_1$ ,  $A_2$ ,  $\{B_{\mu,1}, i = O(1)p_\mu - 1, 3 \leq \mu \leq \tilde{k}\}$  in the expression (63) for  $e_n$  is of the order of magnitude of  $O(h^2)$ . If

$$(76-i) \quad F_j(s-1) = O(1), \quad j = 1, 2, \quad \text{and}$$

$$(76-ii) \quad F_{\mu,1}(s-1) = O(1), \quad i = O(1)p_\mu - 1, \quad \text{for all } s,$$

then since  $\psi_{ts} = O(h)$ , for  $s < t$ ,  $z_n = O(n)$ , and  $t \leq n$ ,

$$P_j(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} F_j(s-1) = O(h^{-1}), \quad \text{for } j = 1, 2, \quad \text{and}$$

$$P_{\mu,1}(n) = -h^2 \sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} \zeta_\mu^{s-1} F_{\mu,1}(s-1) = O(h^{-1}), \quad i = O(1)p_\mu - 1,$$

$$3 \leq \mu \leq \tilde{k}.$$

Suppose the local truncation error for the method is

$$T_n = O(h^3), \text{ provided } r \geq 2, \text{ where } r \text{ is defined earlier.}$$

If the round-off error of the method is

$$(77) \quad R_n = O(h^3),$$

then  $E_n = T_n - R_n = O(h^3)$ . Since  $\psi_{ts} = O(h^2)$ , for  $s < t$ , and  $t \leq n$ , and  $\psi_{tt} = O(1)$ ,  $z_n = O(n)$ , we have

$$\sum_{t=1}^{n-1} z_{n-t} \sum_{s=1}^t \psi_{ts} E_{s-1} = O(1).$$

Then  $e_n$  is bounded as  $h \rightarrow 0$ ,  $n \rightarrow \infty$ , in the larger critical region  $x_n = O(h^{-1/2})$ , provided  $r \geq 2$  with underlying assumptions (75), (76-i), (76-ii) and (77). Hence for a given  $A_2$ -stable usual method, the corresponding perturbed linear multistep method (32) applied to a model initial value problem of the form (71) with  $\text{Re}\sqrt{\lambda} \leq 0$ , the numerical solution  $y_n = y(x_n) + e_n$ , is bounded as  $h \rightarrow 0$ ,  $n \rightarrow \infty$ ,  $x_n = O(h^{-1/2})$ , under the assumptions (75), (76-i), (76-ii) and (77), provided  $r \geq 2$ . Hence the perturbed linear multistep method (32) is  $A_2$ -stable under these assumptions.

## CHAPTER - 3

### ERROR ANALYSIS OF THE OPTIMAL MULTISTEP METHOD FOR BOUNDARY VALUE PROBLEM

#### 3.1 Introduction

In this chapter, we shall study certain theoretical aspects of the  $\beta$ -optimal multistep method, where optimization is done with respect to the  $\beta$ -coefficients, with prefixed  $\alpha$ 's, which can be used for the numerical solution of two point boundary value problems of the form:

$$(1) \quad y'' + f(x, y) = 0, \quad y(a) = A, \quad y(b) = B,$$

where  $-\infty < a < b < \infty$ ,  $A$  and  $B$  are arbitrary constants, and the function  $f(x, y)$  satisfies the following conditions :

(i)  $f(x, y)$  is defined and continuous in the strip  $a \leq x \leq b$ ,  $-\infty < y < \infty$ , where  $a$  and  $b$  are finite.

(ii) There exists a constant  $L$  such that for any  $x \in [a, b]$  and any two numbers  $y$  and  $y^*$

$$| f(x, y) - f(x, y^*) | \leq L | y - y^* |.$$

In addition to satisfying the conditions (i) and (ii),  $f(x, y)$  has continuous partial derivatives with respect to  $y$ . If

$$u_* = \inf_{a \leq x \leq b} \frac{\partial f}{\partial y} \quad \text{and} \quad u^* = \sup_{a \leq x \leq b} \frac{\partial f}{\partial y},$$

then we consider the following two categories of boundary value problems, with either  $-\infty < u^* \leq 0$ , or  $0 < u^* < \pi^2/(b-a)^2$ .

Convergence analysis of a usual multistep method has been

given in Henrici [96], using monotone matrices, for solving a two point boundary value problem of M-type, for which  $\frac{\partial f}{\partial y} \leq 0$ . Here we shall derive the convergence and the stability analysis of the  $\beta$ -optimal method, also using monotone matrices, for solving a two point boundary value problem of M-type, for which

$$-\infty < \frac{\partial f}{\partial y} < \pi^2 / (b-a)^2.$$

For a multistep method of the form

$$(2) \quad \sum_{i=0}^k \alpha_i y_{n-1+i} - h^2 \sum_{i=0}^k \beta_{i,n} f_{n-1+i} = 0; \quad 1 \leq n \leq N-1,$$

where  $f_m = f(x_m, y_m)$ , adopted for the numerical solution of a two point boundary value problem of the form (1), as  $y_0$  and  $y_N$  are being determined by the boundary conditions, the unknowns are  $y_1, y_2, \dots, y_{N-1}$ . If the step number  $k$  of the difference equation is greater than two, new unknown values  $y_{-1}$  or  $y_N$  become introduced, for which there is no equation. This difficulty can be avoided by suitably modifying the difference equations near the boundary points. This situation does not arise, if  $k = 2$ , in (2) and if the associated difference operator has positive order  $p$ , then the difference equation (2) is an equation of the form

$$(3) \quad \sum_{i=0}^2 \alpha_i y_{n-1+i} - h^2 \sum_{i=0}^2 \beta_i f_{n-1+i} = 0,$$

where  $\alpha_0 = -1, \alpha_1 = 2, \alpha_2 = -1$  and  $\beta_0 + \beta_1 + \beta_2 = 1$ . Let  $p$  denote the order of the linear difference operator associated with the difference scheme (3).

If  $\beta_0 = 0, \beta_1 = 1, \beta_2 = 0$  in (3), it becomes Stormer's method with function evaluation at one point, given by

$$(4) \quad -y_{n-1} + 2y_n - y_{n+1} - h^2 f_n = 0,$$

with  $p = 2$ , as the order of the associated difference operator.

If  $\beta_0 = \frac{1}{12}$ ,  $\beta_1 = \frac{10}{12}$ ,  $\beta_2 = \frac{1}{12}$  in (3), it becomes Cowell's method with function evaluation at three points, given by

$$(5) \quad -y_{n-1} + 2y_n - y_{n+1} - h^2 \left\{ \frac{1}{12} f_{n-1} + \frac{10}{12} f_n + \frac{1}{12} f_{n+1} \right\} = 0,$$

$p = 4$ , being the order of the associated difference operator. The  $\beta$ -optimal method corresponding to the schemes (4) and (5) will be the subject of study in this chapter.

In section 3.2, we shall state some known results which are required in the analysis of the following sections. In section 3.3, we shall derive the relation between the  $\beta$ -coefficients of the optimal method and those of the corresponding usual methods (4) and (5). In section 3.4, we shall describe the monotone properties of some matrices which are needed in the analysis of the succeeding sections. In section 3.5, we discuss the convergence analysis of the  $\beta$ -optimal methods corresponding to the usual methods (4) and (5). In section 3.6, we shall derive the order of convergence of a system of nonlinear difference equations with optimal  $\beta$ -coefficients for solving a nonlinear boundary value problem by Newton's method. In section 3.7, we shall discuss the stability analysis of the  $\beta$ -optimal methods and of the corresponding usual methods (4) and (5).

### 3.2 Some Known Results On Linear Multistep Method For BVP

In this section, we shall state some known results on linear multistep methods adopted for the numerical solution of a two point boundary value problem of the form (1).

The linear difference operator associated with the multistep method (2) for solving a boundary value problem (1) is given by

$$L[y(x);h] = \sum_{i=0}^k \alpha_i y(x+ih) + h^2 \sum_{i=0}^k \beta_i y''(x+ih).$$

Let  $L[y(x);h]$  be applied to functions which have continuous derivatives of sufficiently high order. Then expanding by Taylor series, in powers of  $h$ , we obtain

$$L[y(x);h] = C_0 y(x) + C_1 y'(x)h + \dots + C_q y^{(q)}(x)h^q + \dots$$

where the coefficients  $C_q$  ( $q = 1, 2, \dots$ ) are independent of  $y(x)$ .

$$C_0 = \sum_{i=0}^k \alpha_i, \quad C_1 = \sum_{i=0}^k i\alpha_i$$

$$C_q = \frac{1}{q!} \sum_{i=0}^k i^q \alpha_i + \frac{1}{(q-2)!} \sum_{i=0}^k i^{q-2} \beta_i, \quad q = 2, 3, \dots$$

Definition 1: (Henrici [96]) The order  $p$  of the difference operator  $L[y(x);h]$  is defined as the unique integer such that

$$C_q = 0, \quad q = 0, 1, \dots, p+1; \quad C_{p+2} \neq 0.$$

Lemma 1: (Henrici [96]) Let  $L[y(x);h]$  be a difference operator of order  $p > 0$ . There exists a constant  $G > 0$ , depending only on  $L$ , such that

$$|L[y(x);h]| \leq h^{p+2}GY, \quad a \leq x, \quad x+kh \leq b,$$

for all functions  $y(x)$  having a continuous derivative of order  $p+2$  in  $[a,b]$ , where  $Y = \max_{a \leq x \leq b} |y^{(p+2)}(x)|$ .

Theorem 1: (Theorem 7.2, Henrici [96]) Let  $A = (a_{ij})$  be a matrix of order  $n \geq 2$  and  $W$  be the set of the first  $n$  integers. Then  $A$  is irreducible if and only if for any two integers  $i$  and  $j$ ,  $i \in W$ ,  $j \in W$ , there exists a sequence of nonzero elements of  $A$  of the form

$$\{a_{i_1, i_1}, a_{i_1, i_2}, a_{i_2, i_3}, \dots, a_{i_{m-1}, j}\}.$$

Corollary 1: (Corollary of Theorem 7.2, Henrici [96]) A tridiagonal matrix  $A = (a_{ij})$  is irreducible if and only if

$$a_{i, i-1} \neq 0, \quad i = 2, 3, \dots, n; \text{ and } a_{i, i+1} \neq 0, \quad i = 1, 2, \dots, n-1.$$

Let  $z$  be a vector. By the notation  $z \geq 0$ , we mean that all the components  $z_i$  of the vector  $z$ , satisfy  $z_i \geq 0$ . Let us denote the null matrix by  $\underline{0}$ .

Definition 2: (Henrici [96]) A matrix  $A$ , with real elements is called monotone, if  $Az \geq \underline{0}$  implies  $z \geq 0$ .

Theorem 2: (Theorem 7.3, Henrici [96]) A matrix  $A$  is monotone if and only if the elements of the inverse matrix  $A^{-1}$  are non-negative.

Theorem 3: (Theorem 7.2, Henrici [96]) Let the matrix  $A = (a_{ij})$  be irreducible and satisfy the conditions

$$(i) \quad a_{ij} \leq 0, \quad i \neq j; \quad i, j = 1, 2, \dots, n,$$

$$(ii) \quad \sum_{j=1}^n a_{ij} \begin{cases} \geq 0, & i = 1, 2, \dots, n, \\ > 0, & \text{for at least one } i. \end{cases}$$

Then  $A$  is a monotone matrix.

Theorem 4: (Theorem 7.5, Henrici [96]) Let the matrices  $A$  and  $B$  be monotone and assume that

$$A - B \geq \underline{0}$$

Then

$$B^{-1} - A^{-1} \geq \underline{0}.$$

Definition 3: (Definition 1.7, Varga [188]) An  $n \times n$  complex matrix  $A = (a_{ij})$  is diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}| \quad \text{for all } 1 \leq i \leq n.$$

Corollary 2: (Corollary 1 of Theorem 3.11, Varga [188]) If  $A = (a_{ij})$  is a real, irreducibly diagonally dominant  $n \times n$  matrix with  $a_{ij} \leq 0$  for all  $i \neq j$ , and  $a_{ii} > 0$  for all  $1 \leq i \leq n$ , then  $A^{-1} > \underline{0}$ , where  $\underline{0}$  is the null matrix.

We define  $\|v\|$ , a norm of a vector  $v = (v_1, v_2, \dots, v_n)^T$ , by

$$\|v\| = \max_{1 \leq i \leq n} |v_i|$$

and  $\|A\|$ , a norm of a matrix  $A = (a_{ij})_{i,j=1(1)n}$ , by

$$\|A\| = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|.$$



Lemma 2: (Lemma 7.1, Henrici [188]) Let  $A$  be a matrix such that  $\|A\| = k < 1$ , and let  $I$  denote the unit matrix. Then the matrix  $(I - A)^{-1}$  exists, and

$$\|(I - A)^{-1}\| \leq \frac{1}{1 - k}.$$

The following lemma gives an upper bound for the local truncation error of the  $\beta$ -optimal method. The proof follows along the lines of the proof of Lemma 4, in section 2.3.

Lemma 3: Let

$$L[Y(x);h] = \sum_{i=0}^k \alpha_{k-i} Y(x+(k-i)h) + h^2 \sum_{i=0}^k \beta_{k-i} Y''(x+(k-i)h)$$

be the difference operator of a  $k$ -step usual method of order  $p > 0$  and let

$$\hat{L}[Y(x);h] = \sum_{i=0}^k \alpha_{k-i} Y(x+(k-i)h) + h^2 \sum_{i=0}^k \hat{\beta}_{k-i,x} Y''(x+(k-i)h)$$

be the difference operator of the corresponding  $\beta$ -optimal method, where  $\hat{\beta}_{k-i,x}$ ;  $0 \leq i \leq k$ , are the optimal coefficients depending on  $x$ , corresponding to the usual coefficients  $\beta_{k-i}$ ;  $0 \leq i \leq k$ . Let the exact solution  $y(x)$  of the differential equation have a continuous derivative of order  $p+2$  for  $x \in [a,b]$  and let

$$Y = \max_{\substack{a \leq x \leq b \\ 0 \leq i \leq p+2}} |y^{(i)}(x)|.$$

Then in Hilbert space,  $H$ , in which  $D_x^{p+2}$ , the representer of  $p+2$ -th derivative evaluation functional at  $x$  exists and is uniformly bounded on  $[a,b]$

$$|\hat{L}[y(x);h]| \leq h^{p+2} G^* Y, \quad a \leq x, \quad x+kh \leq b,$$

where  $G^*$  is a positive constant.

### 3.3 Relation Between Optimal and Usual $\beta$ -Coefficients

The following two lemmas relate the  $\beta$ -coefficients of the  $\beta$ -optimal multistep method and the corresponding usual multistep method.

Lemma 4: Let

$$(6) \quad \sum_{i=-k}^k \alpha_i y_{n-i} - h^2 \sum_{j=-q}^q \beta_j f_{n-j} = 0, \quad k \leq n \leq N-k,$$

be a consistent usual method of order  $p > 2q$ , where  $\alpha_i$ 's and  $\beta_j$ 's are symmetric in the sense that  $\alpha_i = \alpha_{-i}$  and  $\beta_j = \beta_{-j}$ , for all  $i$ 's and  $j$ 's. If

$$\sum_{i=-k}^k \alpha_i y_{n-i} - h^2 \sum_{j=-q}^q \hat{\beta}_{j,n} f_{n-j} = 0, \quad k \leq n \leq N-k,$$

is the corresponding  $\beta$ -optimal method in a Hilbert space,  $H$ , in which  $D_x^{p+2}$ , the representer of  $p+2$ -th derivative evaluation functional at  $x \in [a,b]$ , exists and is uniformly bounded on  $[a,b]$ , there holds

$$|\hat{\beta}_{j,n} - \beta_j| = o(h^2), \quad h \rightarrow 0, \quad j = -q(1)q,$$

uniformly for every  $n = k(1)N-k$ .

Proof: By the symmetry of the usual coefficients, the equations for  $C_q$ , given in Definition 1, show that actually  $p \geq 2q+2$ . Hence by Lemma 5 of Chapter 2, we have

$$|\hat{\beta}_{j,n} - \beta_j| = o(h^{p-(2q+1)+1}) = o(h^2), \quad h \rightarrow 0, \quad j = -q(1)q,$$

uniformly for every  $n = k(1)N-k$ .

Remark: It is easily seen that at least the highest degree polynomial precision usual method of type (6) has order  $p$  indeed  $\geq 2q+2$ .

It follows from Lemma 4, that in case of Cowell's method (5),

$$|\hat{\beta}_{in} - \beta_i| = \phi_{in} h^2, \quad i = 0, 1, 2; \quad 1 \leq n \leq N-1,$$

where  $\phi_{in}$ ,  $i = 0, 1, 2$ ,  $1 \leq n \leq N-1$  are some constants.

$$\text{Let } \Phi = \max_{1 \leq n \leq N-1} \left| \sum_{i=0}^2 \phi_{in} \right|.$$

$$\text{Then } |(\hat{\beta}_{in} - \beta_i)| \leq \Phi h^2, \quad i = 0, 1, 2; \quad 1 \leq n \leq N-1,$$

where  $\beta_0 = 1/12$ ,  $\beta_1 = 10/12$ ,  $\beta_2 = 1/12$ . Thus we get the inequalities

$$\frac{1}{12} - \Phi h^2 \leq \hat{\beta}_{in} \leq \frac{1}{12} + \Phi h^2, \quad \text{for } i = 0, 2 \quad \text{and}$$

$$\frac{10}{12} - \Phi h^2 \leq \hat{\beta}_{in} \leq \frac{10}{12} + \Phi h^2.$$

$$\text{So, } \hat{\beta}_{in} \geq 0, \quad \text{for } i = 0, 1, 2$$

$$\text{if, } \frac{1}{12} - \Phi h^2 \geq 0 \quad \text{and} \quad \frac{10}{12} - \Phi h^2 \geq 0, \quad \text{i.e. if, } \Phi h^2 \leq \frac{1}{12}.$$

Hence we have the following:

Corollary 3: Assume that  $|\hat{\beta}_{in} - \beta_i| \leq \Phi h^2$ ,  $1 \leq n \leq N-1$ , where  $\hat{\beta}_{in}$  are the optimal coefficients corresponding to the usual coefficients  $\beta_i$ ,  $i = 0(1)2$ , of Cowell's method (5). Then in a Hilbert space,  $H$ , in which  $D_x^6$ , the representer of 6-th derivative evaluation functional at  $x$ , exists and is uniformly bounded on  $[a, b]$ ,  $\hat{\beta}_{in} \geq 0$ ,  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ , if  $\Phi h^2 \leq \frac{1}{12}$ .

It also follows from Lemma 4, that in case of Störmer's method (4), with  $\beta_1 = 1$ ,

$$|\hat{\beta}_{1n} - 1| = \theta_{n,h} h^2, \quad i = 0, 1, 2; \quad 1 \leq n \leq N-1,$$

where  $\theta_{n,h}$ ,  $1 \leq n \leq N-1$  are some constants.

$$\text{Let } \Theta = \max_{1 \leq n \leq N-1} |\theta_{n,h}|.$$

$$\text{Then } |(\hat{\beta}_{1n} - 1)| \leq \Theta h^2, \quad i = 0, 1, 2; \quad 1 \leq n \leq N-1.$$

$$\text{So, } \hat{\beta}_{1n} \geq 0, \quad \text{if } \Theta h^2 \leq 1.$$

Thus we have the following:

Corollary 4: Assume that  $|\hat{\beta}_{1n} - 1| \leq \Theta h^2$ ,  $1 \leq n \leq N-1$ , where  $\hat{\beta}_{1n}$  are the optimal coefficients corresponding to the usual coefficients  $\beta_1 = 1$ , of Stormer's method (4). Then in a Hilbert space,  $H$ , in which  $D_x^4$ , the representer of 4-th derivative evaluation functional at  $x \in [a, b]$ , exists and is uniformly bounded on  $[a, b]$ ,  $\hat{\beta}_{1n} \geq 0$ ,  $1 \leq n \leq N-1$ , if  $\Theta h^2 \leq 1$ .

In the following lemma, we are showing that in  $H^2(C_r)$ -space, and in  $L^2(\hat{C}_r)$ -space, the optimal  $\beta$ -coefficient indeed differ from the corresponding usual  $\beta$ -coefficient of Stormer's method by  $O(h^2)$ , what it is expected, by Lemma 4. However, in  $H_{a,b}^{(3)}$  space, in which  $D_x^4$ , the representer for the 4-th derivative evaluation functional does not exist, this difference is of  $O(h)$ .

Lemma 5 : Let

$$(10) \quad \sum_{i=0}^2 \alpha_i y_{n-1+i} - h^2 \hat{\beta}_{1,n} f_n = 0; \quad 1 \leq n \leq N-1$$

be the  $\beta$ -optimal method, corresponding to Stormer's usual method (4), where  $\hat{\beta}_{1,n}$ ;  $1 \leq n \leq N-1$  are the optimal coefficients corresponding to the usual coefficient  $\beta_1 = 1$ , at the point  $x_n$ .

Then in  $H^2(C_r)$  space and  $L^2(\hat{C}_r)$  space,

$$|\hat{\beta}_{1,n} - \beta_1| = O(h^2), \quad h \rightarrow 0, \text{ uniformly for every } n = 1, 2, \dots, N-1.$$

However, in  $H_{a,b}^{(3)}$  space,

$$|\hat{\beta}_{1,n} - \beta_1| = O(h), \quad h \rightarrow 0, \text{ uniformly for every } n = 1, 2, \dots, N-1.$$

Proof: From the normal equation (11) of Chapter 1, we can obtain

$\hat{\beta}_{1,n}$ , at a point  $x_n$ ,  $1 \leq n \leq N-1$ , which is given by

$$(11) \quad \hat{\beta}_{1,n} = - \frac{1}{h^2} \frac{\sum_{i=0}^2 \alpha_i D2(x_{n-1+i}, \bar{x}_n)}{D2''(x_n, \bar{x}_n)},$$

where  $D2(t, \bar{z}) = \frac{\partial^2}{\partial z^2} K(t, \bar{z})$ ; and  $D2''(t, \bar{z}) = \frac{\partial^2}{\partial t^2} D(t, \bar{z})$ .

Case 1: In  $H^2(C_r)$  space :

$$K(t, \bar{z}) = \frac{1}{\pi} \frac{r}{(r^2 - tz)}, \quad D2(t, \bar{z}) = \frac{1}{\pi} \frac{rt^2}{(r^2 - tz)^3},$$

$$\text{and} \quad D2''(t, \bar{z}) = \frac{1}{\pi} \frac{2r[r^2(r^2 + 4t\bar{z}) + t^2\bar{z}^2]}{(r^2 - tz)^5}.$$

Putting the values of  $K(t, \bar{z})$ ,  $D2(t, \bar{z})$  and  $D2''(t, \bar{z})$  in (11), we

get  $\hat{\beta}_{1,n} = \text{Num} / \text{Din}$ , where

$$\text{Num} = - \frac{1}{h^2} \left[ - \frac{2rx_{n-1}^2}{(r^2 - x_{n-1}x_n)^3} + \frac{4rx_n^2}{(r^2 - x_n^2)^3} - \frac{2rx_{n+1}^2}{(r^2 - x_{n+1}x_n)^3} \right],$$

$$\text{Din} = \frac{4r[r^2(r^2 + 4x_n^2) + x_n^4]}{(r^2 - x_n^2)^5},$$

$$\text{writing } x_n = x, \text{ and } C = \frac{(r^2 - x^2)^2}{2h^2(r^4 + 4r^2x^2 + x^4)},$$

$$\begin{aligned}
\hat{\beta}_{1,n} &= c \left[ \frac{(x-h)^2 (r^2-x^2)^3}{(r^2-x^2+hx)^3} + \frac{(x+h)^2 (r^2-x^2)^3}{(r^2-x^2-hx)^3} - 2x^2 \right] \\
&= c \left[ (x-h)^2 \left\{ 1 + \frac{hx}{r^2-x^2} \right\}^{-3} + (x+h)^2 \left\{ 1 - \frac{hx}{r^2-x^2} \right\}^{-3} - 2x^2 \right] \\
&= c \left[ 2h^2 + 4hx \frac{3hx}{r^2-x^2} + 2(x^2+h^2) \frac{6h^2x^2}{(r^2-x^2)^2} + 4hx \frac{10h^3x^3}{(r^2-x^2)^3} \right. \\
&\quad \left. + 2(x^2+h^2) \frac{15h^4x^4}{(r^2-x^2)^4} + 4hx \frac{21h^5x^5}{(r^2-x^2)^5} + \dots \right] \\
&= 2h^2 c \left[ 1 + \frac{6x^2}{r^2-x^2} + \frac{6x^2(x^2+h^2)}{(r^2-x^2)^2} + \frac{20h^2x^4}{(r^2-x^2)^3} + \frac{15h^2x^4(x^2+h^2)}{(r^2-x^2)^4} \right. \\
&\quad \left. + \frac{42h^4x^6}{(r^2-x^2)^5} + \dots \right] \\
&= \frac{(r^2-x^2)^2}{(r^4+4r^2x^2+x^4)} \left[ \frac{(r^4+4r^2x^2+x^4)}{(r^2-x^2)^2} + \frac{h^2(6r^4x^2+8r^2x^4+x^6)}{(r^2-x^2)^4} + o(h^4) \right] \\
&= 1 + h^2 \frac{(6r^4x^2+8r^2x^4+x^6)}{(r^2-x^2)^2(r^4+4r^2x^2+x^4)} + o(h^4).
\end{aligned}$$

Case 2 : In  $L^2(\hat{C}_r)$  space :

$$K(t, \bar{z}) = \frac{1}{\pi} \frac{r^2}{(r^2 - t\bar{z})^2}, \quad D2(t, z) = \frac{1}{\pi} \frac{6r^2t^2}{(r^2 - t\bar{z})^4},$$

$$\text{and } D2''(t, \bar{z}) = \frac{1}{\pi} \frac{12r^2 [r^2(r^2 + 6t\bar{z}) + 3t^2z^2]}{(r^2 - t\bar{z})^6}.$$

Putting the values of  $K(t, \bar{z})$ ,  $D2(t, \bar{z})$  and  $D2''(t, \bar{z})$  in (11), we get  $\hat{\beta}_{1,n} = \text{Num} / \text{Din}$ . Now

$$\text{Num} = -\frac{1}{h^2} \left[ -\frac{6r^2 x_{n-1}^2}{(r^2 - x_{n-1} x_n)^4} + \frac{12r^2 x_n^2}{(r^2 - x_n^2)^4} - \frac{6r^2 x_{n+1}^2}{(r^2 - x_n x_{n+1})^4} \right],$$

$$\text{Din} = \frac{12r^2 [r^2 (r^2 + 6x_n^2) + 3x_n^4]}{(r^2 - x_n^2)^6}, \quad \text{So that}$$

$$\text{writing } x_n = x, \text{ and } C = \frac{(r^2 - x^2)^2}{2h^2 (r^4 + 6r^2 x^2 + 3x^4)},$$

$$\hat{\beta}_{1,n} = C \left[ \frac{(x-h)^2 (r^2 - x^2)^4}{(r^2 - x^2 + hx)^4} + \frac{(x+h)^2 (r^2 - x^2)^4}{(r^2 - x^2 - hx)^4} - 2x^2 \right]$$

$$= C \left[ (x-h)^2 \left\{ 1 + \frac{hx}{r^2 - x^2} \right\}^{-4} + (x+h)^2 \left\{ 1 - \frac{hx}{r^2 - x^2} \right\}^{-4} - 2x \right]$$

$$= C \left[ 2h^2 + 4hx \frac{4hx}{r^2 - x^2} + 2(x^2 + h^2) \frac{10h^2 x^2}{(r^2 - x^2)^2} + 4hx \frac{20h^3 x^3}{(r^2 - x^2)^3} \right. \\ \left. + 2(x^2 + h^2) \frac{35h^4 x^4}{(r^2 - x^2)^4} + 4hx \frac{56h^5 x^5}{(r^2 - x^2)^5} + \dots \right]$$

$$= 2h^2 C \left[ 1 + \frac{8x^2}{r^2 - x^2} + \frac{10x^2(x^2 + h^2)}{(r^2 - x^2)^2} + \frac{40h^2 x^4}{(r^2 - x^2)^3} + \frac{35h^2 x^4(x^2 + h^2)}{(r^2 - x^2)^4} + \dots \right]$$

$$= \frac{(r^2 - x^2)^2}{(r^4 + 6r^2 x^2 + 3x^4)} \left[ \frac{(r^4 + 6r^2 x^2 + 3x^4)}{(r^2 - x^2)^2} + \frac{5h^2(2r^4 x^2 + 4r^2 x^4 + x^6)}{(r^2 - x^2)^4} + o(h^4) \right]$$

$$= 1 + h^2 \frac{5(2r^4 x^2 + 4r^2 x^4 + x^6)}{(r^2 - x^2)^2 (r^4 + 6r^2 x^2 + 3x^4)} + o(h^4),$$

Case 3 : In  $H_{a,b}^{(3)}$  space :

$$K(t,s)=\begin{cases} \frac{1}{\alpha_0} + \frac{1}{5!} \left( \frac{t-a}{b-a} \right)^5 + \sum_{i=1}^2 \frac{1}{i!^2 \alpha_1} \left( \frac{s-a}{b-a} \right)^i \left[ \left( \frac{t-a}{b-a} \right)^i + \frac{(-1)^{2-i} \alpha_1 i!}{(5-i)!} \left( \frac{t-a}{b-a} \right)^{5-i} \right] \\ \text{for } a \leq t \leq s, \\ \\ \frac{1}{\alpha_0} + \frac{1}{5!} \left( \frac{s-a}{b-a} \right)^5 + \sum_{i=1}^2 \frac{1}{i!^2 \alpha_1} \left( \frac{t-a}{b-a} \right)^i \left[ \left( \frac{s-a}{b-a} \right)^i + \frac{(-1)^{2-i} \alpha_1 i!}{(5-i)!} \left( \frac{s-a}{b-a} \right)^{5-i} \right] \\ \text{for } a \leq t \leq s. \end{cases}$$

$$D2(t,s)=\begin{cases} \frac{1}{(b-a)^2} \left[ \frac{1}{3!} \left( \frac{t-a}{b-a} \right)^3 + \frac{1}{2! \alpha_2} \left( \frac{t-a}{b-a} \right)^2 \right], & \text{for } a \leq t \leq s, \\ \\ \frac{1}{(b-a)^2} \left[ \frac{1}{3!} \left( \frac{s-a}{b-a} \right)^3 - \frac{1}{2!} \left( \frac{s-a}{b-a} \right)^2 \left( \frac{t-a}{b-a} \right) + \frac{1}{2!} \left( \frac{t-a}{b-a} \right)^2 \left( \frac{s-a}{b-a} \right) + \frac{1}{2! \alpha_2} \left( \frac{t-a}{b-a} \right)^2 \right], \\ \text{for } s \leq t \leq b \end{cases}$$

$$D2''(t,s)=\begin{cases} \frac{1}{(b-a)^4} \left[ \left( \frac{t-a}{b-a} \right) + \frac{1}{\alpha_2} \right], & \text{for } a \leq t \leq s, \\ \\ \frac{1}{(b-a)^4} \left[ \left( \frac{s-a}{b-a} \right) + \frac{1}{\alpha_2} \right], & \text{for } s \leq t \leq b, \end{cases}$$

Putting the values of  $K(t,\bar{z})$ ,  $D2(t,\bar{z})$ , and  $D2''(t,\bar{z})$  in (11), and writing  $x_n = x$ , we get

$$\begin{aligned} \text{Num} = & -\frac{1}{h^2} \frac{1}{(b-a)^2} \left[ -\left\{ \frac{1}{3!} \left( \frac{x-h-a}{b-a} \right)^3 + \frac{1}{2! \alpha_2} \left( \frac{x-h-a}{b-a} \right)^2 \right\} \right. \\ & \left. + 2 \left\{ \frac{1}{3!} \left( \frac{x-a}{b-a} \right)^3 + \frac{1}{2! \alpha_2} \left( \frac{x-a}{b-a} \right)^2 \right\} \right. \\ & \left. - \left\{ \frac{1}{3!} \left( \frac{x-a}{b-a} \right)^3 - \frac{1}{2!} \left( \frac{x-a}{b-a} \right)^2 \left( \frac{x+h-a}{b-a} \right) + \frac{1}{2!} \left( \frac{x+h-a}{b-a} \right)^2 \left( \frac{x-a}{b-a} \right) + \frac{1}{2! \alpha_2} \left( \frac{x+h-a}{b-a} \right)^2 \right\} \right] \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{h^2} \frac{(x-a)^2}{(b-a)^5} \left[ \left\{ \frac{(x-a)}{3!} \left( 1 - \frac{h}{x-a} \right)^3 + \frac{(b-a)}{2! \alpha_2} \left( 1 - \frac{h}{x-a} \right)^2 \right\} \right. \\
&\quad \left. - 2 \left\{ \frac{(x-a)}{3!} + \frac{b-a}{2! \alpha_2} \right\} \right. \\
&\quad \left. + \left\{ \frac{(x-a)}{3!} - \frac{(x-a)}{2!} \left( 1 + \frac{h}{x-a} \right) + \frac{(x-a)}{2!} \left( 1 + \frac{h}{x-a} \right)^2 + \frac{(b-a)}{2! \alpha_2} \left( 1 + \frac{h}{x-a} \right)^2 \right\} \right] \\
&= \frac{1}{h^2} \frac{(x-a)^2}{(b-a)^5} \left[ h^2 \left\{ \frac{2}{2! (x-a)} + \frac{2(b-a)}{2! \alpha_2 (x-a)^2} \right\} - \frac{h^3}{3! (x-a)^2} \right] \\
&= \frac{1}{h^2} \frac{(x-a)^2}{(b-a)^5} \left[ \frac{h^2}{(x-a)^2} \left( (x-a) + \frac{1}{\alpha_2} (b-a) \right) - \frac{h^3}{3! (x-a)^2} \right].
\end{aligned}$$

$$D_{in} = \frac{1}{(b-a)^4} \left( \frac{x-a}{b-a} + \frac{1}{\alpha_2} \right) = \frac{1}{(b-a)^5} \left( x-a + \frac{1}{\alpha_2} (b-a) \right).$$

$$\text{Hence, } \hat{\beta}_{1,n} = 1 - \frac{h}{3! \left( x-a + \frac{1}{\alpha_2} (b-a) \right)}.$$

### 3.4 Some Auxiliary Results

In this section, we establish some results required in the following sections. The notations for some specific matrices valid throughout this chapter are as follows:

Definition 4:  $J = (j_{n,m})_{n,m=1}^{N-1}$  is a tridiagonal matrix with

$$\begin{aligned}
j_{n,n-1} &= -1, & \text{for } 2 \leq n \leq N-1; \\
j_{n,n} &= 2, & \text{for } 1 \leq n \leq N-1; \\
j_{n,n+1} &= -1, & \text{for } 1 \leq n \leq N-2.
\end{aligned}$$

Definition 5:  $\hat{D} = \text{diag} \{ \hat{\beta}_{1,1}, \hat{\beta}_{1,2}, \dots, \hat{\beta}_{1,N-1} \},$

where  $\hat{\beta}_{1,n}$ ,  $1 \leq n \leq N-1$  are the optimal  $\beta$ -coefficients at the point  $x_n$  corresponding to the usual  $\beta$ -coefficient  $\beta_1 = 1$  in Störmer's method.

Definition 6:  $\hat{P} = (\hat{p}_{i,j})_{i,j=1}^{N-1}$  is a tridiagonal matrix with

$$\hat{p}_{i,i-1} = \hat{\beta}_{0,i}, \quad \text{for } 2 \leq i \leq N-1;$$

$$\hat{p}_{i,i} = \hat{\beta}_{1,i}, \quad \text{for } 1 \leq i \leq N-1;$$

$$\hat{p}_{i,i+1} = \hat{\beta}_{2,i}, \quad \text{for } 1 \leq i \leq N-1.$$

where  $\hat{\beta}_{i,n}$ ,  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$  are the optimal  $\beta$ -coefficients at the point  $x_n$  corresponding to the usual  $\beta$ -coefficients  $\beta_0 = \frac{1}{12}$ ,  $\beta_1 = \frac{10}{12}$ ,  $\beta_2 = \frac{1}{12}$ , in Cowell's method.

Definition 7:  $U = \text{diag} \{u_1, u_2, \dots, u_{N-1}\}$ ,

where  $u_i = \frac{\partial}{\partial y} f(x_i, w_i + \xi_i[v_i - w_i])$ ,  $0 < \xi_i < 1$ ;  $1 \leq i \leq N-1$ ,  $v_i$  and  $w_i$  are any two net functions.

Definition 8:  $U_* = u_* I$ , where  $u_* = \inf_{a \leq x \leq b} \frac{\partial f}{\partial y}$ .

Definition 9:  $U^* = u^* I$ , where  $u^* = \sup_{a \leq x \leq b} \frac{\partial f}{\partial y}$ .

Definition 10:  $P = (p_{i,j})_{i,j=1}^{N-1}$  is a tridiagonal matrix with

$$p_{i,i} = \frac{10}{12}, \quad \text{for } 1 \leq i \leq N-1;$$

$$p_{i,i-1} = \frac{1}{12}; \quad \text{for } 2 \leq i \leq N-1;$$

$$p_{i,i+1} = \frac{1}{12}; \quad \text{for } 1 \leq i \leq N-2.$$

Definition 11 :  $T = (t_{i,j})_{i,j=1}^{N-1}$  is a tridiagonal matrix with all the components of its tridiagonal part as 1.

$I$  is the identity matrix and  $0$  is the null matrix.

In the case of Cowell's method (5), according to Corollary 3, let  $\hat{\beta}_{1n} - \beta_1 = \phi_{1n} h^2$ , for  $i = 0, 1, 2; 1 \leq n \leq N-1$ .

Definition 12:  $\hat{\phi} = (\hat{\phi}_{ij})_{i,j=1}^{N-1}$  is a tridiagonal matrix, where

$$\hat{\phi}_{i,i-1} = \phi_{0i}, \quad 2 \leq i \leq N-1;$$

$$\hat{\phi}_{ii} = \phi_{1i}, \quad 1 \leq i \leq N-1;$$

$$\hat{\phi}_{i,i+1} = \phi_{2i}, \quad 1 \leq i \leq N-2.$$

and  $\|\hat{\phi}\| = \Phi$ .

In the case of Stormer's method (4), according to Corollary 4, let  $\hat{\beta}_{1n} - 1 = \theta_n h^2$ , for  $1 \leq n \leq N-1$ .

Definition 13:  $\hat{\theta} = \text{diag}\{\theta_1, \theta_2, \dots, \theta_{N-1}\}$ , and  $\|\hat{\theta}\| = \Theta$ .

Let us recall Corollary 3 and Corollary 4 and establish the following lemmas.

Lemma 6 : If  $-\infty < u^* \leq 0$ , then  $J - h^2 \hat{D}U$  is monotone matrix for all  $h$ , provided  $\Theta h^2 \leq 1$ .

Proof :  $J - h^2 \hat{D}U = (s_{nm})_{n,m=1}^{N-1}$  is irreducible, since for  $n \neq m$  such that  $|n-m|=1$ ,  $s_{nm} = -1$  ( $\neq 0$ ), and we have

$$\sum_{m=1}^{N-1} s_{nm} = \begin{cases} 1 - h^2 \hat{\beta}_{1,1} u_1 \geq 1 - h^2 \hat{\beta}_{1,1} u^* > 0; & \text{for } n = 1 \\ 1 - h^2 \hat{\beta}_{1,N-1} u_{N-1} \geq 1 - h^2 \hat{\beta}_{1,N-1} u^* > 0; & \text{for } n = N-1 \\ -h^2 \hat{\beta}_{1,n} u_n \geq -h^2 \hat{\beta}_{1,n} u^* \geq 0; & \text{for } 2 \leq n \leq N-2 \end{cases}$$

provided,  $\hat{\beta}_{1,n} \geq 0$ , for  $1 \leq n \leq N-1$ , which would be the case if  $\Theta h^2 \leq 1$ , by Corollary 4. Hence, by Theorem 3, the lemma follows.

Lemma 7 : If  $0 < u^* < \pi^2/(b-a)^2$ , then  $J - h^2 \hat{D}U$  is monotone matrix for all  $h < H_0$ ; where  $H_0$  is the smallest positive root of the equation:

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

provided  $\Theta h^2 \leq 1$ .

Proof:  $J - h^2 \hat{D}U$  is irreducible, since its off-diagonal elements of the tridiagonal part are -1, that is nonzero. Following Chawla [45], we can write

$$U = U^- + U^+; \text{ where } U^- \leq \underline{0} \text{ and } U^+ > \underline{0}.$$

Then

$$J - h^2 \hat{D}U = J - h^2 \hat{D}(U^- + U^+) = (J - h^2 \hat{D}U^-) - h^2 \hat{D}U^+$$

$$= \hat{M} - h^2 \hat{D}U^+ = \hat{M} (I - h^2 \hat{M}^{-1} \hat{D}U^+), \text{ where } \hat{M} = J - h^2 \hat{D}U^-.$$

By Lemma 6,  $\hat{M}$  is monotone matrix provided  $\Theta h^2 \leq 1$ .  $J - h^2 \hat{D}U$  will be monotone matrix, if  $I - h^2 \hat{M}^{-1} \hat{D}U^+$  is so.  $I - h^2 \hat{M}^{-1} \hat{D}U^+$  will be monotone matrix if its spectral radius,  $\rho(h^2 \hat{M}^{-1} \hat{D}U^+) < 1$ .

Now,  $\hat{M} \geq J$ , if  $\hat{D} \geq \underline{0}$ , which would be the case if  $\Theta h^2 \leq 1$ , according to Corollary 4.  $J$  is also a monotone matrix. Hence, by Theorem 4, we have

$$\hat{M}^{-1} \leq J^{-1}, \text{ if } \Theta h^2 \leq 1.$$

Let us use the spectral radius. Then

$$\rho(h^2 \hat{M}^{-1} \hat{D}U^+) \leq h^2 u^* \rho(J^{-1}) \max_{1 \leq n \leq N-1} |\hat{\beta}_{1,n}|$$

since,  $\rho(J^{-1}) = \frac{1}{4 \sin^2 \frac{\pi h}{2(b-a)}}$ , we get

$$\rho(h^2 \hat{M}^{-1} \hat{D}U^+) \leq \frac{h^2 u^*}{4 \sin^2 \frac{\pi h}{2(b-a)}} (1 + \Theta h^2) = g(h), \text{ say.}$$

$$\lim_{H \rightarrow 0} g(H) = \frac{u^*(b-a)^2}{\pi^2} < 1, \quad \text{and} \quad \lim_{H \rightarrow 2(b-a)} g(H) = \infty.$$

Hence  $g$  assumes the value 1, at some  $h$  in  $(0, 2(b-a))$ . Let  $H_0$  denote the smallest positive root of the equation  $g(H) = 1$ , i.e.,

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

Hence for all  $h < H_0$ ,  $\rho(h^2 \hat{M}^{-1} \hat{D} U^+) < 1$ , and the matrix  $I - h^2 \hat{M}^{-1} \hat{D} U^+$  is monotone. Hence the lemma.

Lemma 8 : If  $-\infty < u^* \leq 0$ , then  $J - h^2 \hat{P} U$  is irreducible and monotone matrix for all  $h < \sqrt{\frac{6}{-u_*}}$ , provided  $\Phi h^2 \leq \frac{1}{12}$ .

Proof.  $J - h^2 \hat{P} U = (t_{nm})_{n,m=1}^{N-1}$  will be an irreducible matrix, if the off diagonal elements of its tridiagonal part are nonzero, i.e. if, for  $n \neq m$ , such that  $|n-m| = 1$ ,  $t_{nm} = -1 - h^2 \hat{\beta}_{i,n} u_m \neq 0$ , for  $i = 0, 2$ ;  $1 \leq m, n \leq N-1$ . Since,  $u_* \leq u_m \leq u^*$ , for  $1 \leq m \leq N-1$ , and  $\frac{1}{12} - \Phi h^2 \leq \hat{\beta}_{i,n} \leq \frac{1}{12} + \Phi h^2$ ; and  $\hat{\beta}_{i,n} \geq 0$ ,  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ , provided  $\Phi h^2 \leq \frac{1}{12}$ , by Corollary 3. Thus we get

$$-h^2 \hat{\beta}_{i,n} u_m \leq -h^2 \hat{\beta}_{i,n} u_* \leq -h^2 \left( \frac{1}{12} + \Phi h^2 \right) u_* \leq \frac{-h^2 u_*}{6};$$

for  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ .

Hence, the condition of irreducibility is satisfied

$$\text{if, } \frac{-h^2 u_*}{6} < 1 \quad \text{i.e., if } h < \sqrt{\frac{6}{-u_*}}.$$

Now, for  $n \neq m$ , such that  $|n-m| = 1$ ,  $t_{n,m} = -1 - h^2 \hat{\beta}_{i,n} u_m \leq 0$ , for  $i = 0, 2$ ;  $1 \leq m, n \leq N-1$ , which is true by the condition of irreducibility. Now

$$\sum_{m=1}^{N-1} t_{n,m} = \begin{cases} -h^2(\hat{\beta}_{0,n} u_{n-1} + \hat{\beta}_{1,n} u_n + \hat{\beta}_{2,n} u_{n+1}) \\ \geq -h^2(\hat{\beta}_{0,n} + \hat{\beta}_{1,n} + \hat{\beta}_{2,n}) u^* \geq 0, & \text{for } 2 \leq n \leq N-1, \\ 1 - h^2(\hat{\beta}_{1,1} u_1 + \hat{\beta}_{2,1} u_2) \geq 1 - h^2(\hat{\beta}_{1,1} + \hat{\beta}_{2,1}) u^* > 0, \\ & \text{for } n = 1, \\ 1 - h^2(\hat{\beta}_{0,N-1} u_{N-2} + \hat{\beta}_{1,N-1} u_{N-1}) \\ \geq 1 - h^2(\hat{\beta}_{0,N-1} + \hat{\beta}_{1,N-1}) u^* > 0, & \text{for } n = N-1. \end{cases}$$

provided  $\hat{\beta}_{i,n} \geq 0$ , for  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ , which holds provided  $\Phi h^2 \leq \frac{1}{12}$ , by Corollary 3. Hence by Theorem 3, the lemma follows.

Lemma 9 Suppose,  $0 < u^* < \pi^2/(b-a)^2$ . If  $u_* < 0$ , then  $J - h^2 \hat{P}U$  is monotone for all  $h < \min \{h_1, h_2\}$ , and if  $u_* > 0$ , then  $J - h^2 \hat{P}U$  is monotone for all  $h < h_2$ , provided  $\Phi h^2 \leq \frac{1}{12}$ , where  $h_1 = \sqrt{\frac{6}{-u_*}}$ , and  $h_2$  is the smallest positive root of the equation :

$$H^2 u^* [(1 + 3\Phi H^2) - (\frac{1}{12} + \Phi H^2) 4\sin^2 \frac{\pi H}{2(b-a)}] - 4\sin^2 \frac{\pi H}{2(b-a)} = 0.$$

Proof:  $J - h^2 \hat{P}U = (t_{nm})_{n,m=1}^{N-1}$  will be an irreducible matrix, if the off-diagonal elements of its tridiagonal part are nonzero, which is the case, if  $-1 - h^2 \hat{\beta}_{i,n} u_m \neq 0$ , for  $i = 0, 2$ ;  $1 \leq m, n \leq N-1$ .

By Corollary 3,  $\frac{1}{12} - \Phi h^2 \leq \hat{\beta}_{i,n} \leq \frac{1}{12} + \Phi h^2$ , and  $\hat{\beta}_{i,n} \geq 0$ , if  $\Phi h^2 \leq \frac{1}{12}$ , for  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ .

Now,  $u_* \leq u_m \leq u^*$ ; for  $1 \leq m \leq N-1$ .

If  $u_* \geq 0$ , then  $u_m \geq 0$ .

So,  $-h^2 \hat{\beta}_{i,n} u_m \leq 0$ , if  $\hat{\beta}_{i,n} \geq 0$ , for  $i = 0, 2$ ;  $1 \leq m, n \leq N-1$ .

Hence,  $J - h^2 \hat{P} U$  is irreducible matrix provided  $\Phi h^2 \leq \frac{1}{12}$ .

If  $u_* < 0$ , then

$$-h^2 \hat{\beta}_{1,n} u_m \leq -h^2 \hat{\beta}_{1,n} u_* \leq -h^2 u_* \left( \frac{1}{12} + \Phi h^2 \right) \leq \frac{-h^2 u_*}{6}$$

for  $i = 0, 2; 1 \leq n \leq N-1$ , provided  $\Phi h^2 \leq \frac{1}{12}$ . Hence, the matrix  $J - h^2 \hat{P} U$  is irreducible,

$$\text{if } \frac{-h^2 u_*}{6} < 1, \quad \text{i.e., if } h < \sqrt{\frac{6}{-u_*}} = h_1, \text{ say.}$$

Now for both the cases when  $u_* \geq 0$ , or  $u_* < 0$ , we proceed as follows.

As in Lemma 7, let  $U = U^- + U^+$ , where  $U^- \leq \underline{0}$  and  $U^+ > 0$ . Then

$$J - h^2 \hat{P} U = \hat{M} - h^2 \hat{P} U^+ = \hat{M}(I - h^2 \hat{M}^{-1} \hat{P} U^+), \text{ where } \hat{M} = J - h^2 \hat{P} U^-.$$

By Lemma 8,  $\hat{M}$  is irreducible and monotone for all  $h < \sqrt{\frac{6}{-u_*}}$ .  $J - h^2 \hat{P} U$  will be a monotone matrix if  $I - h^2 \hat{M}^{-1} \hat{P} U^+$  is so.  $I - h^2 \hat{M}^{-1} \hat{P} U^+$  will be a monotone matrix, if the spectral radius,  $\rho(h^2 \hat{M}^{-1} \hat{P} U^+) < 1$ .

Now,  $\hat{M} \geq J$ , if  $\hat{P} \geq 0$ , which would be the case, if  $\Phi h^2 \leq \frac{1}{12}$ , according to Corollary 3. Hence, by Theorem 4, we have

$$\hat{M}^{-1} \leq J^{-1}, \quad \text{if } \Phi h^2 \leq \frac{1}{12}.$$

Now by the definition of  $\hat{P}$ ,  $P$  and  $T$  from Definitions 6, 10, 11 and using the fact that  $|\beta_{1,n} - \beta_1| \leq \Phi h^2$   $i = 0, 1, 2; 1 \leq n \leq N-1$ , we have

$$\begin{aligned} \hat{P} &\leq P + h^2 \Phi T = \left(I - \frac{1}{12} J\right) + \Phi h^2 (-J + 3I) \\ &= (1 + 3\Phi h^2)I + \left(-\frac{1}{12} - \Phi h^2\right)J. \end{aligned}$$

Since,  $J$  is a monotone matrix,  $J^{-1} \geq \underline{0}$ , by Theorem 2.

$$\text{So, } J^{-1} \hat{P} \leq (1 + 3\Phi h^2)J^{-1} + \left(-\frac{1}{12} - \Phi h^2\right)I$$

Using the spectral radius,

$$\rho(h^2 \hat{M}^{-1} \hat{P} U^+) \leq \rho(h^2 J^{-1} \hat{P} U^+) \leq h^2 u^* \rho(J^{-1} \hat{P}),$$

Using the fact,  $\rho(J^{-1}) = \frac{1}{4 \sin^2 \frac{\pi h}{2(b-a)}}$ , we get

$$\rho(h^2 \hat{M}^{-1} \hat{P} U^+) \leq h^2 u^* \left[ (1 + 3\Phi h^2) \frac{1}{4 \sin^2 \frac{\pi h}{2(b-a)}} + \left(-\frac{1}{12} - \Phi h^2\right) \right] = g_1(h), \text{ say.}$$

$$\lim_{H \rightarrow 0} g_1(H) = \frac{u^*(b-a)^2}{\pi^2} < 1, \text{ and } \lim_{H \rightarrow 2(b-a)} g_1(H) = \infty.$$

Hence  $g_1$  assumes the value 1, at some  $H$  in  $(0, 2(b-a))$ . Let  $h_2$  denote the smallest positive root of the equation  $g_1(H) = 1$ , i.e., of the equation:

$$H^2 u^* \left[ (1 + 3\Phi H^2) - \left(\frac{1}{12} + \Phi H^2\right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

Hence for all  $h < h_2$ ,  $\rho(h^2 \hat{M}^{-1} \hat{P} U^+) < 1$ , and the matrix  $I - h^2 \hat{M}^{-1} \hat{P} U^+$  is monotone. Hence the lemma follows.

Lemma 10 : If  $-\infty < u^* \leq 0$ , then  $J - h^2 \hat{D} U^*$  is monotone matrix for all  $h$ , provided  $\Theta h^2 \leq 1$ .

Proof: Proceeding as in Lemma 6, the result follows.

Lemma 11 : If  $0 < u^* < \pi^2/(b-a)^2$ , then  $J - h^2 \hat{D} U^*$  is monotone matrix for all  $h < H_0$ , where  $H_0$  is the smallest root of the equation :

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

provided  $\Theta h^2 \leq 1$ .

Proof :  $J - h^2 \hat{D} U^*$  is irreducible, since the off diagonal elements of its tridiagonal part are  $-1$ , that is nonzero.



Since,  $U^* > 0$ , we can write,

$$J - h^2 \hat{D}U^* = J(I - h^2 J^{-1} \hat{D}U^*).$$

$J$  being a monotone matrix,  $J - h^2 \hat{D}U^*$  will be a monotone matrix if  $(I - h^2 J^{-1} \hat{D}U^*)$  is so. Proceeding as in Lemma 7,  $(I - h^2 J^{-1} \hat{D}U^*)$  is a monotone matrix for all  $h < H_0$ . Hence the Lemma.

Lemma 12 : If  $-\infty < u^* \leq 0$ , then  $J - h^2 \hat{P}U^*$  is a monotone matrix for all  $h < \sqrt{\frac{6}{-u^*}}$ , provided  $\Phi h^2 \leq \frac{1}{12}$ .

Proof:  $J - h^2 \hat{P}U^* = (t_{nm}^*)_{n,m=1}^{N-1}$  will be an irreducible matrix, if the off-diagonal elements of its tridiagonal part are nonzero, which would be the case, if for  $n \neq m$ , such that  $|n-m| = 1$ ,

$$t_{nm}^* = -1 - h^2 \hat{\beta}_{i,n} u^* \neq 0, \quad i = 0, 2; \quad 1 \leq n \leq N-1.$$

Now, since  $u^* \leq 0$ , and by Corollary 3,  $\hat{\beta}_{i,n} \geq 0$ , if  $\Phi h^2 \leq \frac{1}{12}$ ,

$$-h^2 \hat{\beta}_{i,n} u^* \leq -h^2 u^* \left( \frac{1}{12} + \Phi h^2 \right) \leq \frac{-h^2 u^*}{6}.$$

So, the condition of irreducibility is satisfied

$$\text{if,} \quad -\frac{h^2 u^*}{6} < 1, \quad \text{i.e., if} \quad h < \sqrt{\frac{6}{-u^*}}.$$

For  $n \neq m$ , such that  $|n-m| = 1$ ,  $t_{nm}^* = -1 - h^2 \hat{\beta}_{i,n} u^* \leq 0$ ; for  $i = 0, 2$ , by the condition of irreducibility. Now,

$$\sum_{m=1}^{N-1} t_{n,m}^* = \begin{cases} -h^2 (\hat{\beta}_{0,n} + \hat{\beta}_{1,n} + \hat{\beta}_{2,n}) u^* \geq 0, & \text{for } 2 \leq n \leq N-2; \\ 1 - h^2 (\hat{\beta}_{1,1} + \hat{\beta}_{2,1}) u^* > 0, & \text{for } n = 1; \\ 1 - h^2 (\hat{\beta}_{0,N-1} + \hat{\beta}_{1,N-1}) u^* > 0, & \text{for } n = N-1; \end{cases}$$

as  $\hat{\beta}_{i,n} \geq 0$ , provided  $\Phi h^2 \leq \frac{1}{12}$ . Hence, by Theorem 4, the lemma follows.

Lemma 13 : Suppose,  $0 < u^* < \pi^2/(b-a)^2$ , then  $J - h^2 P U^*$  is a monotone matrix for all  $h < h_2$ , where  $h_2$  is the smallest positive root of the equation:

$$H^2 u^* \left[ (1 + 3\Phi H^2) - \left( \frac{1}{12} + \Phi H^2 \right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

provided  $\Phi h^2 \leq \frac{1}{12}$ .

Proof :  $J - h^2 \hat{P} U^* = (t_{nm}^*)_{n,m=1}^{N-1}$  is irreducible matrix, since the off-diagonal elements of its tridiagonal part are nonzero. By Corollary 3, we have:

$$\frac{1}{12} - \Phi h^2 \leq \hat{\beta}_{1,n} \leq \frac{1}{12} + \Phi h^2 \quad \text{and} \quad \hat{\beta}_{1,n} \geq 0, \quad \text{if } \Phi h^2 \leq \frac{1}{12}.$$

So, 
$$-h^2 \hat{\beta}_{1,n} u^* \leq h^2 u^* (\Phi h^2 - \frac{1}{12}) \leq 0.$$

So, for  $n \neq m$ , such that  $|n - m| \leq 1$ ,  $t_{nm}^* = -1 - h^2 \hat{\beta}_{1,n} u^* \neq 0$ ; for  $i = 0, 2$ ;  $1 \leq n \leq N-1$ . Now, since  $U^* \geq 0$ , we can write,

$$J - h^2 \hat{P} U^* = J(I - h^2 J^{-1} \hat{P} U^*).$$

$J$  being a monotone matrix,  $J - h^2 \hat{P} U^*$  will be a monotone matrix if  $(I - h^2 J^{-1} \hat{P} U^*)$  is monotone. Proceeding as in Lemma 9,  $(I - h^2 J^{-1} \hat{P} U^*)$  is monotone for all  $h < h_2$ , provided  $\Phi h^2 \leq \frac{1}{12}$ . Hence the Lemma.

Lemma 14 : If  $0 < u^* < \pi^2/(b-a)^2$ , then  $J - h^2 U$  and  $J - h^2 U^*$  are monotone matrices for all  $h < H_0$ , where  $H_0$  is the smallest positive root of the equation

$$H^2 u^* - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

In particular, the result holds for all  $h < H_1$ , where

$$H_1 = \sqrt{\frac{24(b-a)^2}{\pi^2} \left( 1 - \sqrt{\frac{u^*(b-a)^2}{\pi^2}} \right)}.$$

Proof :  $J - h^2 U^* = J(I - J^{-1} h^2 U^*)$ , where  $U^* > 0$ .  $J$  is monotone matrix. So it is sufficient to prove that  $(I - J^{-1} h^2 U^*)$  is monotone matrix. The first assertion follows along the lines of the proof of Lemma 7. For the second assertion, we use the inequality  $\sin^2 x \geq x^2(1 - x^2/6)^2$ ,  $x \geq 0$ , which is a slightly sharper form of the inequality used in Chawla [45].

Lemma 15: If  $-\infty < u^* \leq 0$ , then  $J - h^2 PU$  is monotone matrix for all  $h < \sqrt{\frac{12}{-u^*}}$ .

Proof : If  $J - h^2 PU = (\tilde{j}_{nm})_{n,m=1}^{N-1}$ , then

$$\sum_{m=1}^{N-1} \tilde{j}_{n,m} = \begin{cases} -h^2 \left( \frac{1}{12} u_{j-1} + \frac{10}{12} u_j + \frac{1}{12} u_{j+1} \right) \geq 0, & \text{for } 2 \leq n \leq N-1, \\ 1 - h^2 \left( \frac{10}{12} u_2 + \frac{1}{12} u_3 \right) > 0, & \text{for } n = 1, \\ 1 - h^2 \left( \frac{1}{12} u_{N-2} + \frac{10}{12} u_{N-1} \right) > 0, & \text{for } n = N-1. \end{cases}$$

So,  $J - h^2 PU$  will be monotone, if for  $n \neq m$ , such that  $|n - m| = 1$ ,  $\tilde{j}_{n,m} \leq 0$ , i.e., if  $-1 - \frac{1}{12} h^2 u_{1,1\pm 1} \leq -1 - \frac{1}{12} h^2 u^* \leq 0$ ; i.e., if  $h \leq \sqrt{\frac{12}{-u^*}}$ . Hence the lemma.

Lemma 16: If  $-\infty < u^* < 0$ , then  $J - h^2 PU^*$  is monotone matrix for all  $h < \sqrt{\frac{12}{-u^*}}$ .

Proof : If  $J - h^2 PU^* = (\hat{j}_{nm})_{n,m=1}^{N-1}$ , then

$$\sum_{m=1}^{N-1} \hat{j}_{n,m} = \begin{cases} -h^2 u^* \geq 0, & \text{for } 2 \leq n \leq N-1; \\ 1 - h^2 \left( \frac{10}{12} + \frac{1}{12} \right) u^* > 0, & \text{for } n = 1, N-1. \end{cases}$$

So,  $J - h^2 P U^*$  will be monotone if for  $n \neq m$ ,  $\hat{j}_{n,m} \leq 0$ , i.e., if  $-1 - \frac{1}{12} h^2 u^* \leq 0$ ; i.e., if  $h \leq \sqrt{\frac{12}{-u^*}}$ . Hence the lemma.

Lemma 17 : If  $0 < u^* < \pi^2/(b-a)^2$ , then  $J - h^2 P U^*$  is monotone matrix for all  $h < h_0$ , where  $h_0$  is the root of the equation :

$$H^2 u^* \left( \frac{1}{4 \sin^2 \frac{\pi H}{2(b-a)}} - \frac{1}{12} \right) = 1.$$

Proof :  $J - h^2 P U^* = J( I - J^{-1} h^2 P U^* )$ , where  $U^* > 0$ .  $J$  being monotone matrix, it is sufficient to prove that  $I - J^{-1} h^2 P U^*$  is monotone matrix. Proceeding as in Lemma 9, the lemma follows.

Lemma 18: Let  $-\infty < u^* \leq 0$ , and  $h < \sqrt{\frac{12}{-u^*}}$ . If  $(J - h^2 P U^*)^{-1} =$

$(g_{nm})_{n,m=1}^{N-1}$ , then

$$g_{nm} = \begin{cases} \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sinh(n\theta) \cdot \sinh(N-m)\theta}{\sinh(N\theta) \cdot \sinh(\theta)}, & n \leq m; \\ \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sinh(m\theta) \cdot \sinh(N-n)\theta}{\sinh(N\theta) \cdot \sinh(\theta)}, & n \geq m; \end{cases}$$

$$\text{where } \sinh(\theta/2) = \frac{h}{2} \sqrt{\frac{-u^*}{1 + \frac{h^2 u^*}{12}}}.$$

Proof : By Lemma 16,  $J - h^2 P U^*$  is monotone matrix, for  $h < \sqrt{\frac{12}{-u^*}}$ .

So it is invertible and  $(J - h^2 P U^*)^{-1} \geq 0$ , by Theorem 2. Multiplying the  $n$ -th row of  $(J - h^2 P U^*)$  with the  $m$ -th column of  $(J - h^2 P U^*)^{-1}$ , we get

$$(12) \quad \left( -1 - \frac{h^2 u^*}{12} \right) g_{n-1,m} + \left( 2 - \frac{10}{12} h^2 u^* \right) g_{n,m} + \left( -1 - \frac{h^2 u^*}{12} \right) g_{n+1,m}$$

$$= \begin{cases} 0, & \text{for } 1 \leq n \leq m-1 \\ 1, & \text{for } n = m \\ 0, & \text{for } m+1 \leq n \leq N-1 \end{cases} ; \quad \text{where}$$

$$(13) \quad g_{0,m} = g_{N,m} = 0.$$

The characteristic equation for (12) is given by

$$(14) \quad \left(-1 - \frac{h^2 u^*}{12}\right) \xi^2 + \left(2 - \frac{10 h^2 u^*}{12}\right) \xi + \left(-1 - \frac{h^2 u^*}{12}\right) = 0.$$

If  $\xi_1$  and  $\xi_2$  are the two roots of (14), then

$$\xi_1 = \frac{a+b}{c} \quad \text{and} \quad \xi_2 = \frac{a-b}{c}, \quad \text{where}$$

$$a = - \left( 2 - \frac{10}{12} h^2 u^* \right), \quad b = h \left( \frac{2}{3} h^2 u^{*2} - 4u^* \right)^{1/2}, \quad c = 2 \left( -1 - \frac{h^2 u^*}{12} \right).$$

Since  $\xi_1 \cdot \xi_2 = 1$ , we can set  $\xi_1 = e^\theta$ , and  $\xi_2 = e^{-\theta}$ . Then,

$$1 + 2 \sinh^2(\theta/2) = \cosh(\theta) = \frac{e^\theta + e^{-\theta}}{2} = \frac{\xi_1 + \xi_2}{2} = \frac{a}{c}.$$

$$\text{So,} \quad \sinh(\theta/2) = \sqrt{\frac{a-c}{2c}} = \frac{h}{2} \sqrt{\frac{-u^*}{1 + \frac{h^2 u^*}{12}}}.$$

Since,  $\xi_1$  and  $\xi_2$  are the two distinct roots of the equation (14),

$$g_{nm} = R e^{n\theta} + S e^{-n\theta}.$$

Let us assume,

$$(15) \quad g_{nm} = \begin{cases} R_1 e^{n\theta} + S_1 e^{-n\theta}, & n \leq m \\ R_2 e^{n\theta} + S_2 e^{-n\theta}, & n \geq m \end{cases}$$

where  $R_1, S_1, R_2, S_2$  are functions of  $m$  only. Using the condition (13), we get,

$$(16) \quad S_1 = -R_1, \text{ and } S_2 = -R_2 e^{2N\theta},$$

where  $R_1$  and  $R_2$  are the constants which are to be determined such that the definitions of  $g_{nm}$  given in (15) are consistent for  $n = m$  and such that the middle condition of (12) holds. Then

$$(17) \quad R_2 = \frac{R_1 (e^{m\theta} - e^{-m\theta})}{e^{m\theta} - e^{(2N-m)\theta}},$$

and

$$(18) \quad R_1 = \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sinh(N-m)\theta}{2 \sinh(N\theta) \cdot \sinh(\theta)}.$$

Using (16), (17), and (18) in (15) the lemma follows.

Lemma 19: Let  $0 < u^* < \pi^2/(b-a)^2$ , and  $h < h_0$ , where  $h_0$  is the root of the equation :

$$H^2 u^* \left( \frac{1}{4 \sin^2 \frac{\pi H}{2(b-a)}} - \frac{1}{12} \right) = 1.$$

If  $(J - h^2 P U^*)^{-1} = (g_{nm})_{n,m=1}^{N-1}$ , then

$$g_{nm} = \begin{cases} \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sin(n\theta)}{\sin(N\theta)} \frac{\sin(N-m)\theta}{\sin(\theta)}, & n \leq m \\ \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sin(m\theta)}{\sin(N\theta)} \frac{\sin(N-n)\theta}{\sin(\theta)}, & n \geq m; \end{cases}$$

$$\text{where } \sin(\theta/2) = \frac{h}{2} \sqrt{\frac{u^*}{1 + \frac{h^2 u^*}{12}}}.$$

Proof : By Lemma 17,  $J - h^2 P U^*$  is monotone matrix, for all  $h < h_0$ , so it is invertible and  $(J - h^2 P U^*)^{-1} \geq 0$ , by Theorem 2. As in Lemma 18, multiplying the  $n$ -th row of  $(J - h^2 P U^*)$  with the  $m$ -th

column of  $(J - h^2 P U^*)^{-1}$ , we get the equation

$$(19) \quad \left(-1 - \frac{h^2 u^*}{12}\right) g_{n-1,m} + \left(2 - \frac{10}{12} h^2 u^*\right) g_{n,m} + \left(-1 - \frac{h^2 u^*}{12}\right) g_{n+1,m}$$

$$= \begin{cases} 0, & \text{for } 1 \leq n \leq m-1 \\ 1, & \text{for } n = m \\ 0, & \text{for } m+1 \leq n \leq N-1 \end{cases} ; \text{ where}$$

$$(20) \quad g_{0,m} = g_{N,m} = 0,$$

the characteristic equation of which is given by

$$(21) \quad \left(-1 - \frac{h^2 u^*}{12}\right) \xi^2 + \left(2 - \frac{10}{12} h^2 u^*\right) \xi + \left(-1 - \frac{h^2 u^*}{12}\right) = 0.$$

If  $\xi_1$  and  $\xi_2$  are the two roots of (21), then since  $\xi_1 \cdot \xi_2 = 1$ , we can set

$$\xi_1 = e^{i\theta}, \text{ and } \xi_2 = e^{-i\theta}.$$

$$\text{Now, } 1 - 2 \sin^2(\theta/2) = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\xi_1 + \xi_2}{2} = \frac{a}{c},$$

$$\text{where } a = -\left(2 - \frac{10}{12} h^2 u^*\right), \text{ and } c = 2\left(-1 - \frac{h^2 u^*}{12}\right).$$

$$\text{So, } \sin(\theta/2) = \sqrt{\frac{c-a}{2c}} = \frac{h}{2} \sqrt{\frac{u^*}{1 + \frac{h^2 u^*}{12}}}.$$

Since  $\xi_1, \xi_2$  are distinct roots of the equation (21), we can write

$$(22) \quad g_{nm} = \begin{cases} R_1 e^{in\theta} + S_1 e^{-in\theta}, & n \leq m, \\ R_2 e^{in\theta} + S_2 e^{-in\theta}, & n \geq m, \end{cases}$$

$R_1, S_1, R_2, S_2$  being functions of  $m$  only. Using (20), we get,

$$(23) \quad S_1 = -R_1, \text{ and } S_2 = -R_2 e^{2in\theta},$$

where  $R_1$  and  $R_2$  are the constants which are to be determined such that

the definitions of  $g_{nm}$  given in (22) are consistent for  $n = m$  and such that the middle condition of (19) holds. Then

$$(24) \quad R_2 = \frac{R_1 (e^{im\theta} - e^{-im\theta})}{e^{im\theta} - e^{i(2N-m)\theta}},$$

and

$$(25) \quad R_1 = \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sin(N-m)\theta}{2\sin(N\theta) \cdot \sin(\theta)}.$$

Using (23), (24), and (25) in (22) the result follows.

### 3.5 Convergence Analysis

Definition 14: Discretization error, at a point  $x = x_n$ , is the quantity  $e_n = y_n - y(x_n)$ , where  $y_n$  is the exact solution of the finite difference scheme for solving the BVP (1), calculated without round-off error, and  $y(x)$  is the exact solution of the BVP.

Theorem 5: Let

$$\sum_{i=0}^2 \alpha_i y_{n-1+i} - h^2 \sum_{i=0}^2 \beta_i f_{n-1+i} = 0 ; \quad 1 \leq n \leq N-1$$

be the usual Cowell's method with function evaluation at three points, for solving a BVP (1), and

$$\sum_{i=0}^2 \alpha_i y_{n-1+i} - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} f_{n-1+i} = 0 ; \quad 1 \leq n \leq N-1$$

be the corresponding  $\beta$ -optimal method in a Hilbert space,  $H$ . Then in a Hilbert space,  $H$ , in which  $D_x^6$ , the representer of 6-th derivative evaluation functional at  $x$ , exists and is uniformly bounded on  $[a, b]$ , the  $\beta$ -optimal method is convergent.



Proof : Let

$$L [Y(x);h] \equiv \sum_{i=0}^2 \alpha_i Y(x_{n-1+i}) + h^2 \sum_{i=0}^2 \beta_i y''(x_{n-1+i})$$

and

$$\hat{L} [Y(x);h] \equiv \sum_{i=0}^2 \alpha_i Y(x_{n-1+i}) + h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} y''(x_{n-1+i})$$

be the difference operators associated with the above usual and the  $\beta$ -optimal difference equations respectively, for solving the BVP (1). Let

$$(26) \quad \|Y''\| = \max_{a \leq x \leq b} |Y''(x)|.$$

Then in a Hilbert space,  $H$ , in which  $D_x^6$ , the representer of 6-th derivative evaluation functional at  $x$ , exists and is uniformly bounded on  $[a,b]$ , by Lemma 4, and using (26), we get

$$\begin{aligned} |\hat{L} [Y(x);h] - L [Y(x);h]| &= h^2 \left| \sum_{i=0}^2 (\hat{\beta}_{i,n} - \beta_i) y''(x_{n-1+i}) \right| \\ &\leq h^2 \sum_{i=0}^2 |(\hat{\beta}_{i,n} - \beta_i)| |y''(x_{n-1+i})| \\ &\leq h^2 \|Y''\| \sum_{i=0}^2 |(\hat{\beta}_{i,n} - \beta_i)| \\ &\leq h^2 \|Y''\| o(h^2) = o(h^4). \end{aligned}$$

$$\text{Hence,} \quad |\hat{L} [Y(x);h] - L [Y(x),h]| = o(h^4).$$

Since  $L[Y(x);h]$  is consistent,  $|L[Y(x);h]| = o(h^2) = h^2 \delta(h)$ ,

where  $\delta(h) \rightarrow 0$ , as  $h \rightarrow 0$ , so,

$$|\hat{L} [Y(x);h]| \leq |L [Y(x),h]| + o(h^4) = o(h^2) + o(h^4) = o(h^2), \quad \text{so,}$$

$$(27) \quad \left| \hat{L} [Y(x); h] \right| = h^2 \delta(h).$$

It follows that we can write

$$(28) \quad \sum_{i=0}^2 \alpha_i Y(x_{n-1+i}) - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} f(x_{n-1+i}, Y(x_{n-1+i})) = \chi_n h^2 \delta(h),$$

where  $\chi_n$  is some constant, for  $1 \leq n \leq N-1$ , and  $X = \max_{a \leq x \leq b} |\chi_n|$ .

Whereas the optimal method for  $y_n$ , is given by,

$$(29) \quad \sum_{i=0}^2 \alpha_i Y_{n-1+i} - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} f(x_{n-1+i}, Y_{n-1+i}) = 0.$$

Let  $e_n = y_n - Y(x_n)$ ,  $1 \leq n \leq N-1$ , where  $e_0 = e_N = 0$ ,

and  $u_n = -\frac{\partial}{\partial y} f(x_n, y_n + \xi_n e_n)$ , where  $0 < \xi_n < 1$ ;  $1 \leq n \leq N-1$ .

Subtracting (29) from (28), we get the error equation:

$$(30) \quad \sum_{i=0}^2 \alpha_i e_{n-1+i} - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} u_{n-1+i} e_{n-1+i} = \chi_n h^2 \delta(h), \quad 1 \leq n \leq N-1.$$

In matrix notation,

$$(J - h^2 \hat{P}U) e = \hat{X} h^2 \delta(h),$$

so that

$$(31) \quad e = (J - h^2 \hat{P}U)^{-1} \hat{X} h^2 \delta(h),$$

where  $J$ ,  $\hat{P}$ ,  $U$  and  $U^*$  are defined in the Definitions 4, 6, 7 and 9,

$$e = [e_1, e_2, \dots, e_{N-1}]^T \quad \text{and} \quad \hat{X} = [\chi_1, \chi_2, \dots, \chi_{N-1}]^T.$$

Since,  $U \leq U^*$  and  $\hat{P} \geq \underline{0}$ , we have

$$(32) \quad J - h^2 \hat{P}U \geq J - h^2 \hat{P}U^*.$$

Case 1:  $-\infty < u^* \leq 0$

Then  $J - h^2 \hat{P}U \geq J - h^2 \hat{P}U^* \geq J$ .

Now, by Lemma 8,  $J - h^2 \hat{P}U$  is monotone matrix for all  $h < \sqrt{\frac{6}{-u_*}}$ , provided  $\Phi h^2 \leq \frac{1}{12}$ .  $J$  is also a monotone matrix. Hence by Theorem 4,  $(J - h^2 \hat{P}U)^{-1} \leq J^{-1}$ . As in Henrici([96], p. 363),

$$(J^{-1})_{n,m} = \begin{cases} \frac{(N-n)m}{N}, & n \leq m \\ \frac{n(N-m)}{N}, & n \geq m \end{cases}$$

and

$$\|J^{-1}\|_{\infty} = \frac{N^2}{8}$$

from (31),  $|e| \leq J^{-1} |\hat{X}| h^2 \delta(h)$ ,

where  $|v|$  for a vector  $v = (v_1, v_2, \dots, v_n)^T$  is defined by

$$|v| = (|v_1|, |v_2|, \dots, |v_n|)^T.$$

So,  $\|e\|_{\infty} \leq \|J^{-1}\|_{\infty} \|\hat{X}\|_{\infty} h^2 \delta(h)$ .

Hence,  $|e_n| \leq \|e\|_{\infty} \leq \frac{N^2}{8} X h^2 \delta(h) = \frac{(b-a)^2}{8} X \delta(h)$ ,  $1 \leq n \leq N-1$ .

So,  $e_n \rightarrow 0$ , as  $h \rightarrow 0$ . Hence the convergence.

Case 2 :  $0 < u^* < \pi^2/(b-a)^2$

Then  $U^* > 0$ . By Lemma 9 and 13,  $J - h^2 \hat{P}U$  and  $J - h^2 \hat{P}U^*$  are monotone matrices for all  $h < H_5$  where  $H_5 \leq \min\{h_1, h_2\}$ ;

where  $h_1 = \sqrt{\frac{6}{-u_*}}$ , provided  $\Phi h^2 \leq \frac{1}{12}$  and  $h_2$  is the smallest root of the equation:

$$H^2 u^* \left[ (1 + 3\Phi H^2) - \left( \frac{1}{12} + \Phi H^2 \right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

So, by Theorem 4,  $(J - h^2 \hat{P}U)^{-1} \leq (J - h^2 \hat{P}U^*)^{-1}$ .

Hence, from (31),  $|e| \leq (J - h^2 \hat{P}U^*)^{-1} |\hat{X}| h^2 \delta(h)$ .

So, 
$$\|e\|_{\infty} \leq \| (J - h^2 \hat{P}U^*)^{-1} \|_{\infty} \| \hat{X} \|_{\infty} h^2 \delta(h).$$

Now, as in Theorem 11, established in section 3.7, we have

$$(J - h^2 \hat{P}U^*)^{-1} = [I - (J - h^2 PU^*)^{-1} \hat{\phi} h^4 U^*]^{-1} (J - h^2 PU^*)^{-1},$$

where  $\hat{\phi}$  is given by Definition 12. It follows that

$$\|(J - h^2 \hat{P}U^*)^{-1}\|_{\infty} \leq \| [I - (J - h^2 PU^*)^{-1} \hat{\phi} h^4 U^*]^{-1} \|_{\infty} \|(J - h^2 PU^*)^{-1}\|_{\infty}$$

$$\leq d_1 d_2, \text{ say,}$$

$$\text{where } d_1 = \frac{1}{1 - \frac{\Phi h^2 u^*}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}} = O(1)$$

$$\text{and } d_2 = \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} = O(h^{-2})$$

which are obtained latter in Theorem 11. So

$$|e_n| \leq \|e\|_{\infty} \leq d_1 d_2 X h^2 \delta(h) = X \delta(h), \quad 1 \leq n \leq N-1.$$

So,  $e_n \rightarrow 0$ , as  $h \rightarrow 0$ . Hence the convergence.

Theorem 6 : Let

$$-y_{n-1} + 2y_n - y_{n+1} - h^2 f_n = 0, \quad 1 \leq n \leq N-1$$

be Stormer's method for solving a boundary value problem (1) and

$$y_{n-1} + 2y_n - y_{n+1} - h^2 \hat{\beta}_{1,n} f_n = 0, \quad 1 \leq n \leq N-1$$

be the corresponding optimal method where  $\hat{\beta}_{1,n}$ ;  $1 \leq n \leq N-1$  are the optimal coefficients. Then in a Hilbert space,  $H$ , in which  $D_x^4$ , the representer of 4-th derivative evaluation functional at  $x$  exists and is uniformly bounded on  $[a,b]$ , the optimal method is convergent.

Proof : Let

$$L[y(x);h] \equiv \sum_{i=0}^2 \alpha_i y(x_{n-1+i}) + h^2 y''(x_n)$$

and

$$\hat{L}[y(x);h] \equiv \sum_{i=0}^2 \alpha_i y(x_{n-1+i}) + h^2 \hat{\beta}_{1,n} y''(x_n)$$

be the difference operators associated with the above usual and the  $\beta$ -optimal difference equations for solving the BVP (1). Let

$$\|y''\| = \max_{a \leq x \leq b} |y''(x)|.$$

Then as in Theorem 5, and using Lemma 4,

$$\begin{aligned} |\hat{L}[y(x);h] - L[y(x),h]| &= h^2 |(\hat{\beta}_{1,n} - 1)y''(x_n)| \\ &\leq h^2 \|y''(x)\| |(\hat{\beta}_{1,n} - 1)| = o(h^4). \end{aligned}$$

$$\text{So, } |\hat{L}[y(x);h]| - |L[y(x),h]| = o(h^4).$$

Since  $L[y(x);h]$  is consistent,  $|L[y(x);h]| = o(h^2) = h^2 \delta(h)$ , where  $\delta(h) \rightarrow 0$ , as  $h \rightarrow 0$ , so,

$$|\hat{L}[y(x);h]| \leq |L[y(x),h]| + o(h^4) = o(h^2) + o(h^4) = o(h^2).$$

Then, proceeding as in Theorem 5, and with notations therein we get the matrix equation

$$(33) \quad e = (J - h^2 \hat{D}U)^{-1} \hat{X} h^2 \delta(h).$$

Since,  $U \leq U^*$  and  $\hat{D} > 0$ , we have

$$(34) \quad J - h^2 \hat{D}U \geq J - h^2 \hat{D}U^*.$$

Case 1:  $-\infty < u^* \leq 0$ .

Then  $J - h^2 \hat{D}U \geq J - h^2 \hat{D}U^* \geq J$ .

Now by Lemma 6,  $J - h^2 \hat{D}U$  is monotone matrix, for all  $h$ , provided  $\Theta h^2 \leq 1$ .  $J$  is also a monotone matrix. Hence, by Theorem 4,

$$(J - h^2 \hat{D}U)^{-1} \leq J^{-1}.$$

Then as in the Case 1 of Theorem 5,

$$|e_n| \leq \|e\|_{\infty} \leq \frac{(b-a)^2}{8} X \delta(h); \quad 1 \leq n \leq N-1.$$

So,  $e_n \rightarrow 0$ , as  $h \rightarrow 0$ . Hence the convergence.

Case 2 :  $0 < u^* < \pi^2/(b-a)^2$ .

Then  $U^* > 0$ . By Lemmas 7 and 11,  $J - h^2 \hat{D}U$  and  $J - h^2 \hat{D}U^*$  are monotone matrices for all  $h < H_0$ , provided  $\Theta h^2 \leq 1$ , where  $H_0$  is the smallest positive root of the equation :

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

So, from (34), by Theorem 4, we get

$$(J - h^2 \hat{D}U)^{-1} \leq (J - h^2 \hat{D}U^*)^{-1}.$$

Hence, from (33)

$$|e| \leq (J - h^2 \hat{D}U^*)^{-1} \hat{X} |h^2 \delta(h),$$

so that,  $\|e\|_{\infty} \leq \|(J - h^2 \hat{D}U^*)^{-1}\|_{\infty} \|\hat{X}\|_{\infty} h^2 \delta(h)$ .

Now, as in Theorem 10, to be established in section 3.7,

$$(J - h^2 \hat{D}U^*)^{-1} = [I - (J - h^2 U^*)^{-1} \hat{\Theta} h^4 U^*]^{-1} (J - h^2 U^*)^{-1},$$

$$\begin{aligned} \|(J - h^2 \hat{D}U^*)^{-1}\|_{\infty} &\leq \| [I - (J - h^2 U^*)^{-1} \hat{\Theta} h^4 U^*]^{-1} \|_{\infty} \|(J - h^2 U^*)^{-1}\|_{\infty} \\ &\leq c_1 c_2, \text{ say} \end{aligned}$$

$$\text{where } c_1 = \frac{1}{1 - \Theta h^2 u^* \frac{(b-a)^2}{8} \frac{N\Theta}{\sin(N\Theta)} \frac{\Theta}{\sin\Theta}} = O(1)$$

$$c_2 = \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} = O(h^{-2}),$$

So, we get

$$|e_n| \leq \|e\|_\infty \leq c_1 c_2 X h^2 \delta(h) = X \delta(h), \quad 1 \leq n \leq N-1.$$

So,  $e_n \rightarrow 0$ , as  $h \rightarrow 0$ . Hence the convergence.

### 3.6 Newton's Method For Systems Of Nonlinear Equations

Let the  $n$  nonlinear equations

$$\phi_i(y_1, y_2, \dots, y_n) = 0, \quad 1 \leq i \leq n,$$

for the  $n$  unknowns  $y_1, y_2, \dots, y_n$  be written in the vector form as,

$$(35) \quad \phi(y) = 0.$$

Let  $A(y) = (a_{ij}(y))_{i,j=1}^n$  denote the matrix with the elements given by

$$(36) \quad a_{ij}(y) = \frac{\partial \phi_i(y)}{\partial y_j}.$$

Let the vector  $y = y^{(0)}$  be an initial approximation to a solution of the system (35). Assuming  $A(y^{(\nu)})$ 's involved to be invertible, the successive better approximations  $y^{(\nu)}$ ,  $(\nu = 1, 2, \dots)$  are given by

$$(37) \quad y^{(\nu+1)} = y^{(\nu)} - [A(y^{(\nu)})]^{-1} \phi(y^{(\nu)}), \quad \nu = 0, 1, 2, \dots$$

constitute the Newton's method for the solutions of the system of nonlinear equations (35).

Now we state the following important theorem due to Kantorovich (Theorem 7.6, Henrici [96]).

Theorem 7: Assume that the following conditions are satisfied:

(i) For  $y = y^{(0)}$ , the initial approximation, the matrix  $A(y^{(0)})$  has an inverse  $\Gamma_0 = [A(y^{(0)})]^{-1}$  and an estimate for its norm is known,

$$(38) \quad \|\Gamma_0\| \leq B_0,$$

(ii) The vector  $y^{(0)}$  approximately satisfies the system of equations (35) in the sense that

$$(39) \quad \|\Gamma_0 \phi(y^{(0)})\| \leq \eta_0,$$

(iii) In the region defined by the inequality (42) below, the components of the vector  $\phi(y)$  are twice differentiable with respect to the components of  $y$  and satisfy

$$(40) \quad \sum_{j,k=1}^n \left| \frac{\partial^2 \phi_i(y)}{\partial y_j \partial y_k} \right| \leq K, \quad 1 \leq i \leq n$$

(iv) The constants  $B_0$ ,  $\eta_0$ ,  $K$  introduced above satisfy the inequality

$$(41) \quad h_0 = B_0 \eta_0 K \leq \frac{1}{2}.$$

Then the system of equations (35) has a solution  $y^*$  which is located in the cube

$$(42) \quad \|y - y^{(0)}\| \leq N(h_0) \eta_0 = \frac{1 - \sqrt{1-2h_0}}{h_0} \eta_0.$$

Moreover, the successive approximations  $y^{(\nu)}$  defined by (37) exist and converge to  $y^*$ , and the speed of convergence may be estimated by the inequality

$$(43) \quad \|y^{(\nu)} - y^*\| \leq \frac{1}{2^{\nu-1}} (2h_0)^{2^{\nu-1}-1} \eta_0, \quad \nu = 0, 1, 2, \dots$$



Let us now study the convergence of Newton's Method for the system of nonlinear difference equations with optimal  $\beta$ -coefficients. Let

$$(44) \quad \hat{r}(y) \equiv Jy - h^2 \hat{M}f(y) - \hat{a} = 0$$

be the system of nonlinear difference equations with optimal  $\beta$ -coefficients corresponding to the system with usual coefficients,

$$(45) \quad r(y) \equiv Jy - h^2 Mf(y) - a = 0,$$

where  $\hat{M} = \hat{D}$ , when  $M = I$  in the case of Stormer's method (4), with  $\hat{a} = [A, 0, \dots, 0, B]^T = a$ ,

and  $\hat{M} = \hat{P}$ , when  $M = P$  in the case of Cowell's method (5), with

$$\hat{a} = [A - \hat{\beta}_{0,1} h^2 f(x_0, A), 0, \dots, 0, B - \hat{\beta}_{2, N-1} h^2 f(x_N, B)]^T,$$

$$a = [A - \beta_{0,1} h^2 f(x_0, A), 0, \dots, 0, B - \beta_{2, N-1} h^2 f(x_N, B)]^T,$$

$$f(y) = [f(x_1, y_1), f(x_2, y_2), \dots, f(x_{N-1}, y_{N-1})]^T.$$

The linearized system (44) then reads

$$(46) \quad \hat{r}(y^{(0)}) + [J - h^2 \hat{B}F(y^{(0)})] \Delta y = 0,$$

where  $F(y) = \text{diag} \{f_y(x_1, y_1), f_y(x_2, y_2), \dots, f_y(x_{N-1}, y_{N-1})\}$

$y^{(0)}$  being an initial approximation to a solution of the system (35), we get

$$\hat{r}(y^{(0)}) + \hat{A}(y^{(0)}) \Delta y = 0,$$

where  $\hat{A}(y) = J - h^2 \hat{B} F(y)$ .

Its solution is given by

$$\Delta y = \Delta y^{(0)} = - [\hat{A}(y^{(0)})]^{-1} \hat{r}(y^{(0)}),$$

which implies,  $y^{(1)} = y^{(0)} - [\hat{A}(y^{(0)})]^{-1} \hat{r}(y^{(0)})$ .

To prove the uniqueness of the solution of (44), for solving the BVP (1), suppose, if possible,  $y$  and  $z$  are any two solutions of (44), then we can write

$$f(x_n, z_n) - f(x_n, y_n) = u_n(z_n - y_n),$$

where  $u_n$  is a value of  $\frac{\partial f}{\partial y}$ .

Let  $d_n = z_n - y_n$ ,  $1 \leq n \leq N-1$ ,  $d = [d_1, d_2, \dots, d_{N-1}]^T$  and

$U = [u_1, u_2, \dots, u_{N-1}]^T$ , which is given by Definition 7. Then from the system of equations (44) we get an equation for  $d$ ,

$$(47) \quad [J - h^2 \hat{M}U]d = \underline{0}.$$

If  $[J - h^2 \hat{M}U]^{-1}$  is a nonsingular matrix, then  $d = \underline{0}$  and  $y = z$ .

Now, we shall verify that under certain hypotheses, the conditions of Theorem 7 are satisfied for the system (44) with  $\phi(y) = \hat{r}(y)$  and the solution of (44) is unique.

Firstly, we shall consider the situation, when  $\hat{M} = \hat{P}$  in (44).

If  $\hat{r}(y) = [\hat{r}_1(y), \hat{r}_2(y), \dots, \hat{r}_{N-1}(y)]^T$ ,

then  $\phi_n(y) = \hat{r}_n(y) = -y_{n-1} + 2y_n - y_{n+1}$

$$+ h^2 \{ \hat{\beta}_{0n} f(x_{n-1}, y_{n-1}) + \hat{\beta}_{1n} f(x_n, y_n) + \hat{\beta}_{2n} f(x_{n+1}, y_{n+1}) \}.$$

Let  $L_2 = \max_{\substack{a \leq x \leq b \\ -\infty < y < \infty \\ 1 \leq i, j \leq N-1}} \left| \frac{\partial^2}{\partial y_i \partial y_j} f(x, y) \right|$ , then

using Corollary 3, and the fact that  $\beta_0 + \beta_1 + \beta_2 = \frac{1}{12} + \frac{10}{12} + \frac{1}{12} = 1$ ,

$$\sum_{i, m=1}^{N-1} \left| \frac{\partial^2 \phi_n(y)}{\partial y_i \partial y_m} \right| \leq h^2 L_2 \{ \hat{\beta}_{0n} + \hat{\beta}_{1n} + \hat{\beta}_{2n} \} \leq h^2 L_2 (1 + 3\Phi h^2) \leq \frac{5}{4} h^2 L_2,$$

provided that  $\Phi h^2 \leq \frac{1}{12}$ .

Hence the condition (iii) of Theorem 7, is satisfied with

$$(48) \quad K = \frac{5}{4} h^2 L_2.$$

Let the initial approximations  $y^{(0)}$  be defined by

$$y_n^{(0)} = z(x_n), \quad 1 \leq n \leq N-1.$$

Let

$$(49) \quad \tilde{R} = \max_{1 \leq n \leq N-1} \left| \frac{1}{h^2} \left\{ \sum_{i=0}^2 \alpha_i z(x_{n-1+i}) - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} f(x_{n-1+i}, z(x_{n-1+i})) \right\} \right|.$$

$$\text{If } \hat{r}(y^{(0)}) = \left( \hat{r}_1(y^{(0)}), \hat{r}_2(y^{(0)}), \dots, \hat{r}_{N-1}(y^{(0)}) \right)^T,$$

then for  $1 \leq n \leq N-1$ ,

$$|\hat{r}_n(y^{(0)})| = \left| \sum_{i=0}^2 \alpha_i z(x_{n-1+i}) - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} f(x_{n-1+i}, z(x_{n-1+i})) \right| \leq h^2 \tilde{R}.$$

$$\text{So, } \|\Gamma_0 \hat{r}(y^{(0)})\| \leq \|\Gamma_0\| \|\hat{r}(y^{(0)})\| \leq B_0 h^2 \tilde{R}.$$

Hence, the condition (ii) of Theorem 7, holds with

$$(50) \quad \eta_0 = B_0 h^2 \tilde{R}.$$

$$\text{Let } u_* = \inf_{\substack{a \leq x \leq b \\ 1 \leq i, j \leq N-1}} \frac{\partial f_1}{\partial y_j}, \quad u^* = \sup_{\substack{a \leq x \leq b \\ 1 \leq i, j \leq N-1}} \frac{\partial f_1}{\partial y_j}, \quad U_* = u_* I, \quad U^* = u^* I.$$

$$\text{Then, } \hat{A}(y) = J - h^2 \hat{P}F(y) \geq J - h^2 \hat{P}U^*.$$

Case 1:  $-\infty < u^* \leq 0$

Then  $U^* \leq \underline{0}$ . Since  $F(y^{(0)}) \leq U^* \leq \underline{0}$ ,

$$\hat{A}(y^{(0)}) = J - h^2 \hat{P}F(y^{(0)}) \geq J - h^2 \hat{P}U^* \geq J.$$

Using Lemma 8,  $J - h^2 \hat{P}F(y^{(0)})$  is monotone for all  $h < h_1$ , where

$h_1 = \sqrt{\frac{6}{-u_*}}$ , provided  $\Phi h^2 \leq 1/12$ .  $J$  is also a monotone matrix. Hence by Theorem 4,

$$[\hat{A}(Y^{(0)})]^{-1} \leq J^{-1}.$$

Since, by Henrici [96],  $\|J^{-1}\|_{\infty} \leq \frac{N^2}{8} = \frac{(b-a)^2}{8h^2}$ ,

the condition (i) of Theorem 7, holds with

$$(51) \quad B_0 = \frac{(b-a)^2}{8h^2}.$$

By (50),  $\eta_0 = \frac{(b-a)^2}{8} \tilde{R}$ .

The condition (iv) of Theorem 7, which guarantees the convergence of Newton's process, thus turns out to be satisfied, if

$$h_0 = B_0 \eta_0 K = \frac{(b-a)^2}{8h^2} \frac{(b-a)^2}{8} \tilde{R} \frac{5h^2 L_2}{4} \leq \frac{1}{2},$$

i.e., if

$$(52) \quad \tilde{R} \leq \frac{128}{5} \frac{1}{L_2 (b-a)^4}.$$

Now by Lemma 8,  $J - h^2 \hat{P}U$  is a monotone matrix for all  $h < h_1$ , provided  $\Phi h^2 \leq \frac{1}{12}$  and hence is nonsingular, by Theorem 2. So, by (47), it follows that  $d = \underline{0}$ , which follows that the system of equations  $Jy - h^2 \hat{P}f(y) - \hat{a} = \underline{0}$  has unique solution.

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

Then  $U^* \geq \underline{0}$ . Since  $F(Y^{(0)}) \leq U^*$ ,

$$\hat{A}(Y^{(0)}) = J - h^2 \hat{P}F(Y^{(0)}) \geq J - h^2 \hat{P}U^*.$$

Using Lemma 9,  $J - h^2 \hat{P}F(Y^{(0)})$  is monotone for all  $h < \min\{h_1, h_2\}$ , provided  $\Phi h^2 \leq \frac{1}{12}$ , where  $h_1 = \sqrt{\frac{6}{-u_*}}$ , and  $h_2$  is the smallest

positive root of the equation:

$$H^2 u^* \left[ (1 + 3\Phi H^2) - \left( \frac{1}{12} + \Phi H^2 \right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

So, by Theorem 4,

$$[\hat{A}(Y^{(0)})]^{-1} \leq [J - h^2 \hat{P} U^*]^{-1}.$$

Then

$$(J - h^2 \hat{P} U^*) = J - h^2 (P + h^2 \hat{\phi}) U^* = (J - h^2 P U^*) - h^4 \hat{\phi} U^*,$$

where  $P$  and  $\hat{\phi}$  are the matrices given by Definition 10, 12.

$$\begin{aligned} \text{So, } (J - h^2 \hat{P} U^*)^{-1} &= \left[ (J - h^2 P U^*) [I - (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*] \right]^{-1} \\ &= [I - (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*]^{-1} (J - h^2 P U^*)^{-1}. \end{aligned}$$

So,

$$\begin{aligned} \|(J - h^2 \hat{P} U^*)^{-1}\|_{\infty} &\leq \| [I - (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*]^{-1} \|_{\infty} \|(J - h^2 P U^*)^{-1}\|_{\infty} \\ &\leq d_1 d_2, \text{ say,} \end{aligned}$$

$$\text{where } d_1 = \frac{1}{1 - \frac{\Phi h^2 u^*}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}} = O(1)$$

$$\text{and } d_2 = \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} = O(h^{-2}),$$

which are obtained in Theorem 11 in section 3.7. So, the condition (i) of Theorem 7 holds with

$$(53) \quad B_0 = d_1 d_2.$$

$$\text{By (50),} \quad \eta_0 = d_1 d_2 h^2 \tilde{R}.$$

Note that the required  $K$ , for the Kantorovich Theorem is given by (48). Hence, the condition (iv) of Theorem 7, which guarantees the

convergence of Newton's process, will be satisfied if

$$h_0 = B_0 \eta_0 K = d_1^2 d_2^2 h^2 \tilde{R} \frac{5}{4} h^2 L_2 \leq \frac{1}{2},$$

i.e., if

$$(54) \quad \tilde{R} \leq \frac{128}{5L_2(b-a)^4} \left[ 1 + \frac{h^2 u^*}{12} \right]^2 \left[ \frac{\sin(N\theta)}{N\theta} \frac{\sin\theta}{\theta} - \frac{\Phi h^2 u^*}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \right]^2.$$

Now by Lemma 9,  $J - h^2 \hat{P}U$  is a monotone matrix for all  $h < \min \{h_1, h_2\}$ , provided  $\Phi h^2 \leq \frac{1}{12}$ , where  $h_1$  and  $h_2$  are as given earlier. Hence, it is nonsingular, by Theorem 2. So, by (47), it follows that the system of equations  $Jy - h^2 \hat{P}f(y) - \hat{a} = \underline{0}$  has unique solution.

Thus, we can state the following Theorem.

Theorem 8 : Let  $\hat{r}(y) \equiv Jy - h^2 \hat{P} f(y) - \hat{a} = \underline{0}$ , be the system of finite difference equations with optimal  $\beta$ - coefficients, arisen from a boundary value problem (1). Assume that  $|\hat{\beta}_{i_n} - \beta_i| \leq \Phi h^2$ , for  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ , in a Hilbert space,  $H$ , in which  $D_x^6$ , the representer of 6-th derivative evaluation functional at  $x$  exists and is uniformly bounded in  $[a, b]$ . Let the initial approximation be given by  $y_n^{(0)} = z(x_n)$ ,  $1 \leq n \leq N-1$ , and  $\tilde{R}$  be defined by (49). If  $L_2$  is an upper bound for  $\frac{\partial^2 f(x, y)}{\partial y_i \partial y_j}$  in  $a \leq x \leq b$ ,  $-\infty < y < \infty$ ,  $1 \leq i, j \leq N-1$ , then the system (44) possesses a unique solution which can be obtained by Newton's method, if  $\tilde{R}$  satisfies (52), in case of  $-\infty < u^* \leq 0$ , for all  $h < h_1$ , and (54), in case of  $0 < u^* < \pi^2/(b-a)^2$ , for all  $h < \min \{h_1, h_2\}$ , where  $h_1 = \sqrt{\frac{6}{-u^*}}$ , and  $h_2$  is the smallest positive root of the equation :

$$H^2 u^* \left[ (1 + 3\Phi H^2) - \left( \frac{1}{12} + \Phi H^2 \right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

provided  $\Phi h^2 \leq 1/12$ .

Following the proof of the convergence of Newton's method in the usual case, as given in Henrici [96], let the initial approximations  $y^{(0)}$  be defined by

$$(55) \quad y_n^{(0)} = z(x_n), \quad 1 \leq n \leq N-1,$$

where  $z(x)$  is a  $p+2$  times differentiable function satisfying  $z(a) = A$ ,  $z(b) = B$ ;  $p$  being the order of the difference operator.

Let

$$Z = \max_{a \leq x \leq b} |z^{(p+2)}(x)|$$

and

$$(56) \quad R = \max_{a \leq x \leq b} |z''(x) + f(x, z(x))|.$$

Let  $L[y(x); h]$  and  $\hat{L}[y(x); h]$  be the difference operators associated with the Cowell's usual method (5) and the corresponding  $\beta$ -optimal method (6) for solving the BVP (1).

Let

$$\|y''\| = \max_{a \leq x \leq b} |y''(x)|.$$

Then

$$|\hat{L}[z(x); h]| = \left| \sum_{i=0}^2 \alpha_i z(x+(i-1)h) + h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} z''(x+(i-1)h) \right| = O(h^4).$$

Now, for  $1 \leq n \leq N-1$

$$\begin{aligned} |\hat{r}_n(y^{(0)})| &= \left| \sum_{i=0}^2 \alpha_i z(x_{n-1+i}) - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} f(x_{n-1+i}, z(x_{n-1+i})) \right| \\ &\leq \left| \sum_{i=0}^2 \alpha_i z(x_{n-1+i}) - h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} z''(x_{n-1+i}) \right| \\ &\quad + h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} |z''(x_{n-1+i}) + f(x_{n-1+i}, z(x_{n-1+i}))| \end{aligned}$$

$$\leq |\hat{L}[z(x_n); h]| + h^2 R(\hat{\beta}_{0n} + \hat{\beta}_{1n} + \hat{\beta}_{2n}).$$

Since the order of  $L[y(x); h]$  is  $p=4$ , using Lemma 3 and Corollary 3, we get

$$|\hat{r}_n(y^{(0)})| \leq G^* Zh^6 + h^2 R(1 + 3\Phi h^2) \leq G^* Zh^6 + \frac{5}{4} h^2 R,$$

provided that  $\Phi h^2 \leq \frac{1}{12}$ .

$$\text{So, } \|\Gamma_0 \hat{r}(y^{(0)})\| \leq \|\Gamma_0\| \|\hat{r}(y^{(0)})\| \leq B_0 \left( G^* Zh^6 + \frac{5}{4} h^2 R \right).$$

Hence, the condition (ii) of Theorem 7, holds with

$$(57) \quad \eta_0 = B_0 \left( G^* Zh^6 + \frac{5}{4} h^2 R \right),$$

with  $K$  as in the previous theorem, given by (48).

Case 1:  $-\infty < u^* \leq 0$ .

Let  $B_0$  be as in the Case 1 of the previous Theorem, given by (51).

Using (57) for  $\eta_0$ , the condition (iv) of Theorem 7 is satisfied if

$$h_0 = B_0 \eta_0 K \leq \frac{1}{2},$$

$$\text{i.e., if } h_0 = \frac{(b-a)^2}{8h^2} \frac{(b-a)^2}{8} \left( G^* Zh^4 + \frac{5}{4} R \right) \frac{5}{4} h^2 L_2 \leq \frac{1}{2},$$

i.e., if

$$(58) \quad \left( G^* Zh^4 + \frac{5}{4} R \right) \leq \frac{128}{5} \frac{1}{L_2 (b-a)^4}.$$

The uniqueness of the solution of (44) holds for  $h < h_1$ , as in the Case 1 of the previous theorem, provided  $\Phi h^2 \leq \frac{1}{12}$ .

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

With  $B_0$  as in the Case 2 of the previous Theorem given by (53),

using (57) for  $\eta_0$ , the condition (iv) of Theorem 7 is satisfied, if

$$h_0 = B_0 \eta_0 K = d_1^2 d_2^2 \left( G^* Zh^6 + \frac{5}{4} h^2 R \right) \frac{5}{4} h^2 L_2 \leq \frac{1}{2}, \quad \text{i.e., if}$$



$$(59) \quad \left( G^* Z h^4 + \frac{5}{4} R \right) \leq$$

$$\frac{128}{5L_2(b-a)^4} \left[ 1 + \frac{h^2 u^*}{12} \right]^2 \left[ \frac{\sin(N\theta)}{N\theta} \frac{\sin\theta}{\theta} - \frac{\Phi h^2 u^*}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \right]^2.$$

The uniqueness of the solution of (44) holds for  $h < \min \{h_1, h_2\}$ , as in the Case 2 of the previous theorem, provided  $\Phi h^2 \leq \frac{1}{12}$ .

Thus we get the following corollary.

Corollary 5: Let the system of finite difference equations (44) correspond to a BVP (1). If  $p$  denotes the order of the difference operator  $L[y(x);h]$  associated with the usual method (5), assume that the exact solution  $y(x)$  has a continuous  $(p+2)$ -nd derivative in  $[a,b]$  and let

$$Z = \max_{a \leq x \leq b} |y^{(p+2)}(x)|.$$

Let  $R$  be defined by (56) and the initial approximation be defined by (55) and  $L_2$  is an upper bound for  $\frac{\partial^2 f(x,y)}{\partial y_1 \partial y_j}$  in  $a \leq x \leq b$ ,  $-\infty < y < \infty$ ,  $1 \leq i, j \leq N-1$ . Assume that  $|\hat{\beta}_{i,n} - \beta_i| \leq \Phi h^2$ , for  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ , in a Hilbert space,  $H$ , in which  $D_x^6$ , the representer of 6-th derivative evaluation functional exists and is uniformly bounded in  $[a,b]$ . Then the system (44) possesses a unique solution which can be found by Newton's method, if  $h$  satisfies (58) in case of  $-\infty < u^* \leq 0$ , with  $h < h_1$ , and (59) in case of  $0 < u^* < \pi^2/(b-a)^2$ , with  $h < \min \{h_1, h_2\}$ , where  $h_1 = \sqrt{\frac{6}{-u^*}}$ , and  $h_2$  is the smallest positive root of the equation :

$$H^2 u^* \left[ (1 + 3\Phi H^2) - \left( \frac{1}{12} + \Phi H^2 \right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

provided that  $\Phi h^2 \leq 1/12$ .

Secondly, we shall consider the situation, when  $\hat{M} = \hat{D}$  in (44). Then we have the system of equations  $\hat{r}(y) \equiv Jy - h^2 \hat{D}f(y) - \hat{a} = \underline{0}$ .

If  $\hat{r}(y) = [\hat{r}_1(y), \hat{r}_2(y), \dots, \hat{r}_{N-1}(y)]^T$ ,

then  $\phi_n(y) \equiv \hat{r}_n(y) = -y_{n-1} + 2y_n - y_{n+1} + h^2 \hat{\beta}_{1n} f(x_n, y_n)$ .

Let 
$$L_2 = \max_{\substack{a \leq x \leq b \\ -\infty < y < \infty}} \left| \frac{\partial^2}{\partial y_1 \partial y_j} f(x, y) \right|.$$

Now using Corollary 4, and assuming that  $\Theta h^2 \leq 1$ ,

$$\sum_{1, m=1}^{N-1} \left| \frac{\partial^2 \phi_n(y)}{\partial y_1 \partial y_m} \right| \leq h^2 L_2 \hat{\beta}_{1n} \leq h^2 L_2 (1 + \Theta h^2) \leq 2h^2 L_2.$$

Hence the condition (iii) of Theorem 7, is satisfied with

$$(60) \quad K = 2h^2 L_2.$$

Let the initial approximation  $y^{(0)}$  be defined by

$$y^{(0)} = z(x_n), \quad 1 \leq n \leq N-1, \quad \text{and}$$

$$(61) \quad \tilde{R} = \max_{1 \leq n \leq N-1} \left| \frac{1}{h^2} \left\{ \sum_{i=0}^2 \alpha_i z(x_{n-1+i}) - h^2 \hat{\beta}_{1n} f(x_n, z(x_n)) \right\} \right|.$$

Then for  $1 \leq n \leq N-1$ ,

$$|\hat{r}_n(y^{(0)})| = \left| \sum_{i=0}^2 \alpha_i z(x_{n-1+i}) - h^2 \hat{\beta}_{1n} f(x_n, z(x_n)) \right| \leq h^2 \tilde{R}.$$

$$\text{So,} \quad \|\Gamma_0 \hat{r}_n(y^{(0)})\| \leq \|\Gamma_0\| \|\hat{r}_n(y^{(0)})\| \leq B_0 h^2 \tilde{R}.$$

Hence the condition (iii) of Theorem 7, holds with

$$(62) \quad \eta_0 = B_0 h^2 \tilde{R}.$$

Case 1:  $-\infty < u^* \leq 0$ .

Then  $U^* \leq \underline{0}$ . Since  $F(y^{(0)}) \leq U^*$ ,

$$\hat{A}(y^{(0)}) = J - h^2 \hat{D}F(y^{(0)}) \geq J - h^2 \hat{D}U^* \geq J.$$

By Lemma 6,  $J - h^2 \hat{D}F(y^{(0)})$  is monotone matrix for all  $h$  provided  $\Theta h^2 \leq 1$ .  $J$  is also a monotone matrix. So, by Theorem 2,

$$[J - h^2 \hat{D}F(y^{(0)})]^{-1} \leq J^{-1}.$$

Proceeding as in the Case 1 of Theorem 8, we get,

$$(63) \quad B_0 = \frac{(b-a)^2}{8h^2}.$$

$$\text{By (62) and (63),} \quad \eta_0 = \frac{(b-a)^2}{8} \tilde{R}.$$

and  $K$  is given by (60). The condition (iv) of Theorem 7, which guarantees the convergence of Newton's process, thus turns out to be satisfied, if

$$h_0 = B_0 \eta_0 K = \frac{(b-a)^2}{8h^2} \frac{(b-a)^2}{8} \tilde{R} 2h^2 L_2 \leq \frac{1}{2},$$

i.e., if

$$(64) \quad \tilde{R} \leq \frac{16}{L_2 (b-a)^4}.$$

By Lemma 6,  $J - h^2 \hat{D}U$  is a monotone matrix for all  $h$ , provided  $\Theta h^2 \leq 1$ . Hence by (47), the system of equations  $Jy - h^2 \hat{D}f(y) - \hat{a} = \underline{0}$  has unique solution.

Case 2:  $0 < u^* < \pi^2 / (b-a)^2$ ,

Then  $U^* \geq \underline{0}$ . Since  $F(y^{(0)}) \leq U^*$ ,

$$\hat{A}(y^{(0)}) = J - h^2 \hat{D}F(y^{(0)}) \geq J - h^2 \hat{D}U^*.$$

By Lemma 7,  $J - h^2 \hat{D}F(y^{(0)})$  is a monotone matrix for all  $h < H_0$ , provided  $\Theta h^2 \leq 1$ , where  $H_0$  is the smallest root of the equation:

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

So, by Theorem 4,

$$[\hat{A} (Y^{(0)})]^{-1} \leq [J - h^2 \hat{D}U^*]^{-1}.$$

$$\text{Now, } (J - h^2 \hat{D}U^*) = J - h^2 (I + h^2 \hat{\theta})U^* = (J - h^2 U^*) - h^4 \hat{\theta}U^*,$$

where  $\hat{\theta}$  is given in Definition 13, and by Corollary 4,  $\|\hat{\theta}\| \leq \theta$ .

$$\begin{aligned} \text{So, } (J - h^2 \hat{D}U^*)^{-1} &= \left[ (J - h^2 U^*) [I - (J - h^2 U^*)^{-1} h^4 \hat{\theta}U^*] \right]^{-1} \\ &= [I - (J - h^2 U^*)^{-1} h^4 \hat{\theta}U^*]^{-1} (J - h^2 U^*)^{-1}, \end{aligned}$$

so that

$$\begin{aligned} \|(J - h^2 \hat{D}U^*)^{-1}\|_{\infty} &\leq \| [I - (J - h^2 U^*)^{-1} \hat{\theta} h^4 U^*]^{-1} \|_{\infty} \|(J - h^2 U^*)^{-1}\|_{\infty} \\ &\leq c_1 c_2, \text{ say} \end{aligned}$$

$$\text{where } c_1 = \frac{1}{1 - \theta h^2 u^* \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}} = O(1),$$

$$c_2 = \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} = O(h^{-2}),$$

which are obtained in Theorem 10 in section 3.7. So, the condition (i) of Theorem 7, holds with

$$(65) \quad B_0 = c_1 c_2.$$

$$\text{By (62) and (65), } \eta_0 = c_1 c_2 h^2 \tilde{R}.$$

K is given by (60). Hence, the condition (iv) of Theorem 7, which guarantees convergence of Newton's process, will be satisfied, if

$$h_0 = B_0 \eta_0 K = c_1^2 c_2^2 h^2 \tilde{R}^2 L_2 \leq \frac{1}{2},$$

i.e., if

$$(66) \quad \tilde{R} \leq \frac{16}{L_2(b-a)^4} \left[ \frac{\sin(N\theta)}{N\theta} \frac{\sin\theta}{\theta} - \Theta h^2 u^* \frac{(b-a)^2}{8} \right]^2.$$

By Lemma 7,  $J - h^2 \hat{D}U$  is a monotone matrix for all  $h < H_0$ , where  $H_0$  is as given above, provided  $\Theta h^2 \leq 1$ . Hence, by (47), the system of equations  $Jy - h^2 \hat{D}f(y) - \hat{a} = \underline{0}$  has unique solution.

Thus we can state the following theorem.

Theorem 9 : Let  $\hat{r}(y) \equiv Jy - h^2 \hat{D} f(y) - \hat{a} = \underline{0}$  be the system of finite difference equations with optimal  $\beta$  coefficient, corresponding to usual Stormer's method (4), arisen from a BVP (1). Assume that  $|\hat{\beta}_{i,n} - \beta_i| \leq \Phi h^2$ , for  $i = 0, 1, 2$ ;  $1 \leq n \leq N-1$ , in a Hilbert space,  $H$ , in which  $D_x^4$ , the representer of 4-th derivative evaluation functional at  $x$  exists and is uniformly bounded in  $[a, b]$ . Let the initial approximation be given by  $y_n^{(0)} = z(x_n)$ ,  $1 \leq n \leq N-1$ , and let  $\tilde{R}$  be defined by (61). If  $L_2$  is an upper bound for  $\frac{\partial^2 f(x, y)}{\partial y_i \partial y_j}$  in  $a \leq x \leq b$ ;  $-\infty < y < \infty$ ;  $1 \leq i, j \leq N-1$ , then the above system possesses a unique solution which can be obtained by Newton's method, if  $\tilde{R}$  satisfies (64) in the case of  $-\infty < u^* \leq 0$ , and (66) in the case of  $0 < u^* < \pi^2/(b-a)^2$ , with  $h < H_0$ , where  $H_0$  is the smallest positive root of the equation:

$$H^2 u^* (1 + \Theta h^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

provided  $\Theta h^2 \leq 1$ .

Let us now proceed as in Corollary 5. Let the initial approximation  $y^{(0)}$  be defined by

$$y^{(0)} = z(x_n), \quad 1 \leq n \leq N-1,$$

where  $z(x)$  is a  $p+2 = 4$  times differentiable function satisfying  $z(a) = A$ ,  $z(b) = B$ ;  $p$  being the order of the difference operator  $L[y(x); h]$  and let

$$Z = \max_{a \leq x \leq b} |z^{(4)}(x)|,$$

and 
$$R = \max_{a \leq x \leq b} |z''(x) + f(x, z(x))|.$$

Let  $L[y(x); h]$  and  $\hat{L}[y(x); h]$  be the difference operators associated with Stormer's usual method (4) and the corresponding  $\beta$ -optimal method (10). Since  $L[y(x); h]$  has order  $p = 2$ , by Lemma 3

$$|\hat{L}[z(x); h]| = \left| \sum_{i=0}^2 \alpha_i z(x+(i-1)h) - h^2 \hat{\beta}_{1,n} z''(x) \right| \leq G^* Z h^4.$$

Now, for  $1 \leq n \leq N-1$ ,

$$\begin{aligned} |\hat{r}_n(y^{(0)})| &= \left| \sum_{i=0}^2 \alpha_i z(x_{n-1+i}) - h^2 \hat{\beta}_{1,n} f(x_n, z(x_n)) \right| \\ &\leq |\hat{L}[z(x_n); h]| + h^2 R \hat{\beta}_{1,n} \\ &\leq G^* Z h^4 + h^2 R(1 + \Theta h^2) \leq G^* Z h^4 + 2h^2 R, \end{aligned}$$

provided that  $\Theta h^2 \leq 1$ . So,

$$\|\Gamma_0 \hat{r}(y^{(0)})\| \leq \|\Gamma_0\| \|\hat{r}(y^{(0)})\| \leq B_0 (G^* Z h^4 + 2h^2 R).$$

Hence, the condition (ii) of Theorem 7, holds with

$$(67) \quad \eta_0 = B_0 (G^* Z h^4 + 2h^2 R).$$

Let  $K$  be given by (60).

Case 1:  $-\infty < u^* \leq 0$ .

With  $B_0$  as in the Case 1 of Theorem 9, given by (63). From (67),

$$\eta_0 = \frac{(b-a)^2}{8h^2} (G^* Z h^4 + 2h^2 R).$$

The condition (iv) of Theorem 7, is satisfied, if

$$h_0 = B_0 \eta_0 K = \frac{(b-a)^2}{8h^2} \frac{(b-a)^2}{8h^2} \left( G^* Zh^4 + 2h^2 R \right) 2h^2 L_2 \leq \frac{1}{2},$$

i.e., if

$$(68) \quad \left( G^* Zh^2 + 2R \right) \leq \frac{16}{L_2 (b-a)^4}.$$

The uniqueness of the solution holds as in Case 1 of Theorem 9.

Case 2:  $0 < u^* < \pi^2 / (b-a)^2$ .

$B_0$  is as in Case 2 of Theorem 9, given by (65). Then from (67)

$$\eta_0 = c_1 c_2 \left( G^* Zh^4 + 2h^2 R \right).$$

The condition (iv) of Theorem 7, is satisfied, if

$$h_0 = B_0 \eta_0 K = c_1^2 c_2^2 \left( G^* Zh^4 + 2h^2 R \right) 2h^2 L_2 \leq \frac{1}{2},$$

i.e., if

$$(69) \quad \left( G^* Zh^4 + 2h^2 R \right) \leq \frac{16}{L_2 (b-a)^4} \left[ \frac{\sin(N\theta)}{N\theta} \frac{\sin\theta}{\theta} - \theta h^2 u^* \frac{(b-a)^2}{8} \right]^2.$$

The uniqueness of the solution follows as in Case 2 Theorem 9. Thus we get the following corollary.

Corollary 6: Let the system of finite difference equations  $\hat{r}(y) \equiv Jy - h^2 \hat{D}f(y) - \hat{a} = \underline{0}$  correspond to the BVP (1). Assume that the exact solution  $y(x)$  has a continuous 4-th derivative in  $[a, b]$  and let

$$Z = \max_{a \leq x \leq b} |y^{(4)}(x)|.$$

Assume that  $|\hat{\beta}_{1n} - 1| \leq \theta h^2$ , for  $1 \leq n \leq N-1$ , in a Hilbert space,  $H$ , in which  $D_x^4$ , the representer of 4-th derivative evaluation functional exists and is uniformly bounded in  $[a, b]$ . If the initial approximation  $y^{(0)}$  is defined by (55) and  $R$  is defined (56) and  $L_2$

is an upper bound for  $\frac{\partial^2 f(x, y)}{\partial y_i \partial y_j}$  in  $a \leq x \leq b$ ,  $-\infty \leq y \leq \infty$ ,  $1 \leq i, j \leq N-1$ , then the above system of equations possesses a unique solution, which can be found by Newton's method for all values of  $h$ , provided  $\Theta h^2 \leq 1$ , satisfying (68) in the case of  $-\infty < u^* \leq 0$ , and (69) in the case of  $0 < u^* < \pi^2/(b-a)^2$ , with  $h < H_0$ , where  $H_0$  is the smallest possible root of the equation:

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

### 3.7 Stability Analysis

In this section, we shall establish stability theory for the  $\beta$ -optimal methods, corresponding to the usual Stormer's method (4) and the Cowell's method (5), and for those two usual methods adopted for solving a two point BVP (1). An important application of stability is in obtaining the error estimate which gives a relation between discretization error and local truncation error.

For solving an IVP or a BVP numerically, the set of discrete points  $\{x_j\}$ , employed on the given interval  $[a, b]$ , is called a net. We will take this net to be uniformly spaced, say

$$x_0 = a, \quad x_j = a + jh; \quad j = 1(1)N, \quad h = (b-a)/N.$$

The quantity  $h$  is called net spacing. A rule which assigns to each point  $x_j$  of the net a corresponding  $n$  vector  $v_j$  is called a net function,  $\{v_j\}$ .

Theorem 10: Let  $\hat{L}_1$  be the linear difference operator associated with the optimal method corresponding to the usual Stormer's method



(4) and a net function  $\{v_i\}$  be defined as

$$\hat{L}_1 v_i \equiv - \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} - \hat{\beta}_{1,i} f(x_i, v_i) = 0; \quad 1 \leq i \leq N-1,$$

where  $\hat{\beta}_{1,i}$ ,  $1 \leq i \leq N-1$  are the optimal coefficients corresponding to the usual coefficient  $\beta_1 = 1$  and

$$|\hat{\beta}_{1,i} - 1| \leq \Theta h^2, \quad 1 \leq i \leq N-1.$$

Assume that  $f(x, y)$  has continuous derivatives with respect to  $y$ ,

$$\text{satisfying} \quad u_* \leq \frac{\partial f}{\partial y} \leq u^*.$$

Then  $\hat{L}_1$  is stable in the sense that for all net functions  $\{v_i\}, \{w_i\}$

$$|v_i - w_i| \leq M \left\{ \max \left( |v_0 - w_0|, |v_N - w_N| \right) + \max_{1 \leq j \leq N-1} |\hat{L}_1 v_j - \hat{L}_1 w_j| \right\},$$

for  $1 \leq i \leq N-1$ , for some constant  $M$  and some suitable restriction on  $h$  given in the following two cases.

Case 1 :  $-\infty < u^* < 0$ .

Then  $\hat{L}_1$  is stable with  $M = \hat{M}_1 = C_1 M_1$ , where

$$\hat{M}_1 = \frac{1}{1 - \Theta h^2} \max \left\{ 1, \frac{1}{-u^*} \right\}, \quad \text{for all } h \text{ satisfying } \Theta h^2 < 1.$$

Case 2:  $0 < u^* < \pi^2 / (b-a)^2$ .

Then  $\hat{L}_1$  is stable with  $M = \hat{M}_2 = C_2 M_2$ , where

$$\hat{M}_2 = \frac{1}{1 - \Theta h^2 u^* \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}}.$$

$$\max \left\{ \frac{1}{\cos \frac{N\theta}{2}}, \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \right\},$$

for all  $h < \min \{H_1, H_2\}$ , where  $H_1$  is the smallest positive root of

the equation:

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

and  $H_2$  is the smallest positive root of the equation

$$H^2 u^* - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

provided  $\Theta h^2 < \min \left\{ 1, \frac{8}{(b-a)^2 u^*} \right\}$ .

Proof: If  $\{v_i\}$  and  $\{w_i\}$  are any two net functions, then

$$h^2 \hat{L}_1 v_i = - (v_{i-1} - 2v_i + v_{i+1}) - h^2 \hat{\beta}_{1,1} f(x_i, v_i),$$

$$h^2 \hat{L}_1 w_i = - (w_{i-1} - 2w_i + w_{i+1}) - h^2 \hat{\beta}_{1,1} f(x_i, w_i).$$

$$\begin{aligned} \text{So, } h^2 [\hat{L}_1 v_i - \hat{L}_1 w_i] &= - (v_{i-1} - w_{i-1}) + 2(v_i - w_i) - (v_{i+1} - w_{i+1}) \\ &\quad - h^2 \hat{\beta}_{1,1} (v_i - w_i) \frac{\partial}{\partial y} f(x_i, w_i + \xi_i (v_i - w_i)). \end{aligned}$$

$$\text{Let } u_i = \frac{\partial}{\partial y} f(x_i, w_i + \xi_i (v_i - w_i)), \quad 1 \leq i \leq N-1.$$

Then the above equation becomes

$$\begin{aligned} h^2 [\hat{L}_1 v_i - \hat{L}_1 w_i] &= - (v_{i-1} - w_{i-1}) + (2 - h^2 \hat{\beta}_{1,1} u_i) (v_i - w_i) - (v_{i+1} - w_{i+1}); \\ &\quad \text{for } 1 \leq i \leq N-1. \end{aligned}$$

In matrix notation this system of equations becomes,

$$(70) \quad (J - h^2 \hat{D}U) [v - w] = b_1 + b_2,$$

where  $J$ ,  $\hat{D}$ ,  $U$  are  $(N-1) \times (N-1)$  matrices given in Definitions 4, 5, 7.

$$[v - w] = [(v_1 - w_1), (v_2 - w_2), \dots, (v_{N-1} - w_{N-1})]^T,$$

$$b_1 = [(v_0 - w_0), 0, \dots, 0, (v_N - w_N)]^T,$$

$$b_2 = h^2 [(\hat{L}_1 v_1 - \hat{L}_1 w_1), (\hat{L}_1 v_2 - \hat{L}_1 w_2), \dots, (\hat{L}_1 v_{N-1} - \hat{L}_1 w_{N-1})]^T.$$

Since  $(J - h^2 \hat{D}U)$  is a monotone matrix, by Lemmas 6 and 7, for  $-\infty < u^* \leq 0$  and  $0 < u^* < \pi^2/(b-a)^2$ , respectively. Hence by Theorem 2,  $(J - h^2 \hat{D}U)$  is nonsingular. Hence,

$$(71) \quad [v - w] = (J - h^2 \hat{D}U)^{-1} (b_1 + b_2).$$

Now  $u_* \leq u \leq u^*$ , so that  $U_* \leq U \leq U^*$ ,

where  $U, U_*, U^*$  are the matrices given in Definitions 7, 8, 9. Hence

$$(J - h^2 \hat{D}U) \geq (J - h^2 \hat{D}U^*).$$

If both of these matrices are monotone then by Theorem 4,

$$(72) \quad (J - h^2 \hat{D}U)^{-1} \leq (J - h^2 \hat{D}U^*)^{-1}.$$

Using (72) in (70),

$$(73) \quad [v - w] \leq (J - h^2 \hat{D}U^*)^{-1} (b_1 + b_2).$$

According to Lemma 5, writing  $\hat{\beta}_{1n} - 1 = \theta_n h^2$ ,  $1 \leq n \leq N-1$ , we get

$$(J - h^2 \hat{D}U^*) = J - h^2 (I + h^2 \hat{\theta}) U^* = (J - h^2 U^*) - h^4 \hat{\theta} U^*,$$

where  $\hat{\theta} = \text{diag}\{\theta_1, \theta_2, \dots, \theta_{N-1}\}$  and by Corollary 4,  $\|\hat{\theta}\| \leq \theta$ . If

$$(74) \quad \|(J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*\| \leq k < 1,$$

then by Lemma 2, the matrix  $[I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1}$  exists and

$$\|[I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1}\|_{\infty} \leq \frac{1}{1-k}.$$

So,  $(J - h^2 \hat{D}U^*)^{-1} = [I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1} (J - h^2 U^*)^{-1}.$

Hence from (73), we get

$$[v - w] \leq [I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1} (J - h^2 U^*)^{-1} (b_1 + b_2).$$

Hence for  $1 \leq i \leq N-1$ ,

$$(75) \quad |v_1 - w_1| \leq \| [v - w] \|_{\infty} \leq \| [I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1} \|_{\infty} \\ \cdot \left( \| (J - h^2 U^*)^{-1} b_1 \|_{\infty} + \| (J - h^2 U^*)^{-1} b_2 \|_{\infty} \right).$$

$$\text{Let } J - h^2 U^* = (\hat{d}_{nm})_{n,m=1}^{N-1}, \quad \text{and} \quad (J - h^2 U^*)^{-1} = (d_{nm})_{n,m=1}^{N-1}.$$

Now we shall analyze the situation separately for the two cases.

Case 1:  $-\infty < u^* < 0$ .

By Lemmas 6 and 10,  $J - h^2 \hat{D}U$  and  $J - h^2 \hat{D}U^*$  are monotone for all  $h$ , provided  $\theta h^2 \leq 1$ . Hence for these  $h$ , the inequality (72) holds good. Now,  $J - h^2 U^*$  is irreducibly diagonally dominant with nonpositive off-diagonal elements and positive diagonal elements. Hence by Corollary 2,  $(J - h^2 U^*)^{-1} > \underline{0}$ , i.e.,  $d_{nm} > 0$ , for  $1 \leq n, m \leq N-1$ . By Lemma 3.1 of Baboo [7],

$$(J - h^2 U^*)_{n,m}^{-1} = d_{nm} = \begin{cases} \frac{\sinh(n\theta) \cdot \sinh(N-m)\theta}{\sinh(N\theta) \cdot \sinh\theta}; & n \leq m \\ \frac{\sinh(m\theta) \cdot \sinh(N-n)\theta}{\sinh(N\theta) \cdot \sinh\theta}; & n \geq m \end{cases}$$

$$\text{where } \sinh(\theta/2) = \frac{h}{2} \sqrt{-u^*}.$$

Now we shall calculate  $\| (J - h^2 U^*)^{-1} \|_{\infty}$ .

Let  $a_m$  denote the sum of the elements in the  $m$ -th row of the matrix  $J - h^2 U^*$ . Then, for  $1 \leq n \leq N-1$ ,

$$\sum_{m=1}^{N-1} d_{nm} a_m = \sum_{m=1}^{N-1} d_{nm} \sum_{l=1}^{N-1} \hat{d}_{ml} = \sum_{l=1}^{N-1} \left( \sum_{m=1}^{N-1} d_{nm} \hat{d}_{ml} \right) = \sum_{l=1}^{N-1} \delta_{nl} = 1.$$

Since, for  $1 \leq m \leq N-1$ ,  $a_m \geq -h^2 u^*$ , we have

$$-h^2 u^* \sum_{m=1}^{N-1} d_{nm} \leq \sum_{m=1}^{N-1} d_{nm} a_m = 1, \quad \text{for } 1 \leq n \leq N-1.$$

So,

$$\sum_{m=1}^{N-1} d_{nm} \leq \frac{1}{-h^2 u^*}, \quad \text{for } 1 \leq n \leq N-1.$$

Hence,

$$(76) \quad \| (J - h^2 U^*)^{-1} \|_{\infty} = \max_{1 \leq n \leq N-1} \sum_{m=1}^{N-1} d_{nm} \leq \frac{1}{-h^2 u^*}.$$

Hence by Corollary 4,

$$\begin{aligned} \| (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^* \|_{\infty} &\leq \| (J - h^2 U^*)^{-1} \|_{\infty} h^4 \| \hat{\theta} \|_{\infty} \| U^* \|_{\infty} \\ &\leq \frac{1}{-h^2 u^*} h^4 \theta (-u^*) = \theta h^2 = k, \text{ say.} \end{aligned}$$

Thus, as stated in (74), if  $\theta h^2 < 1$ , then the matrix

$$[I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1} \text{ exists and}$$

$$(77) \quad \| [I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1} \|_{\infty} \leq \frac{1}{1 - \theta h^2} = c_1, \text{ say.}$$

Let  $(J - h^2 U^*)^{-1} b_1 = [\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_{N-1}]^T,$

where  $\tilde{v}_i = (v_0 - w_0) d_{i1} + (v_N - w_N) d_{i,N-1}, \quad 1 \leq i \leq N-1.$

Then  $\| (J - h^2 U^*)^{-1} b_1 \|_{\infty} \leq \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\} \| \tilde{W} \|_{\infty},$

where  $\tilde{W} = [\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_{N-1}]^T$ , with  $\tilde{w}_i = d_{i1} + d_{i,N-1}, \quad 1 \leq i \leq N-1.$

$$\begin{aligned} \text{So, } \tilde{w}_1 &= \frac{\sinh(N-i)\theta}{\sinh(N\theta)} + \frac{\sinh(i\theta)}{\sinh(N\theta)} = \frac{\sinh(N-i)\theta + \sinh(i\theta)}{\sinh(N\theta)} \\ &= \frac{2 \sinh \frac{N\theta}{2} \cdot \cosh \left( \frac{N-2i}{2} \theta \right)}{2 \sinh \frac{N\theta}{2} \cdot \cosh \frac{N\theta}{2}} = \frac{\cosh \left( \frac{N-2i}{2} \theta \right)}{\cosh \frac{N\theta}{2}} \leq 1, \end{aligned}$$

since,  $\cosh$  is a monotonically increasing function in  $[0, \infty)$ . So,

$$(78) \quad \| (J - h^2 U^*)^{-1} b_1 \|_{\infty} \leq \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\}.$$

Again, using (76),

$$\begin{aligned} \| (J - h^2 U^*)^{-1} b_2 \|_\infty &\leq \| (J - h^2 U^*)^{-1} \|_\infty \| b_2 \|_\infty \\ &\leq \frac{1}{-h^2 u^*} h^2 \max_{1 \leq j \leq N-1} | \hat{L}_1 v_j - \hat{L}_1 w_j |. \end{aligned}$$

Hence,

$$(79) \quad \| (J - h^2 U^*)^{-1} b_2 \|_\infty \leq \frac{1}{-u^*} \max_{1 \leq j \leq N-1} | \hat{L}_1 v_j - \hat{L}_1 w_j |.$$

Using (77), (78) and (79) in (75) the result of Case 1 follows.

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

By Lemmas 7 and 11,  $J - h^2 \hat{D}U$  and  $J - h^2 \hat{D}U^*$  are monotone matrices for all  $h < H_1$ , provided  $\Theta h^2 \leq 1$  where  $H_1$  is the smallest positive root of the equation:

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

Hence for these  $h$ , the inequality (72) holds good. Again by Lemma 14,  $J - h^2 U^*$  is a monotone matrix for all  $h < H_2$ , where  $H_2$  is the smallest positive root of the equation:

$$H^2 u^* - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

Then by Theorem 2,  $(J - h^2 U^*)^{-1} \geq 0$ , i.e.,  $d_{nm} \geq 0$ . By Lemma 3.2 of Chawla [45],

$$(J - h^2 U^*)_{n,m}^{-1} = d_{nm} = \begin{cases} \frac{\sin(n\theta) \cdot \sin(N-m)\theta}{\sin(N\theta) \cdot \sin\theta}; & n \leq m \\ \frac{\sin(m\theta) \cdot \sin(N-n)\theta}{\sin(N\theta) \cdot \sin\theta}; & n \geq m, \end{cases}$$

where  $\sin(\theta/2) = \frac{h}{2} \sqrt{u^*}$ .

Now we shall calculate  $\| (J - h^2 U^*)^{-1} \|_\infty$ .

$$\begin{aligned}
\sum_{m=1}^{N-1} |d_{nm}| &= \sum_{m=1}^n d_{nm} + \sum_{m=n+1}^{N-1} d_{nm} \\
&= \sum_{m=1}^n \frac{\sin(m\theta) \cdot \sin(N-n)\theta}{\sin(N\theta) \cdot \sin\theta} + \sum_{m=n+1}^{N-1} \frac{\sin(n\theta) \cdot \sin(N-m)\theta}{\sin(N\theta) \cdot \sin\theta} \\
&\leq \sum_{m=1}^n \frac{m\theta \cdot (N-n)\theta}{\sin(N\theta) \cdot \sin\theta} + \sum_{m=n+1}^{N-1} \frac{n\theta \cdot (N-m)\theta}{\sin(N\theta) \cdot \sin\theta} \\
&\leq \frac{(N-n)\theta \cdot \frac{n(n+1)\theta}{2} + n\theta \cdot \frac{(N-n-1)(N-n)\theta}{2}}{\sin(N\theta) \cdot \sin\theta} \leq \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}.
\end{aligned}$$

So,

$$(80) \quad \| (J - h^2 U^*)^{-1} \|_{\infty} \leq \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}.$$

Hence,  $\| (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^* \|_{\infty} \leq h^4 \| (J - h^2 U^*)^{-1} \| \|\hat{\theta}\| \|U^*\|$ , so that

$$(81) \quad \| (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^* \|_{\infty} \leq \Theta h^2 u^* \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} = k, \text{ say.}$$

Thus as stated in (74), if  $k < 1$ , i.e., if

$$(82) \quad \Theta h^2 < \frac{8}{(b-a)^2 u^*},$$

then the matrix  $[I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1}$  exists and

$$(83) \quad \|[I - (J - h^2 U^*)^{-1} h^4 \hat{\theta} U^*]^{-1}\|_{\infty} \leq \frac{1}{1-k} = c_2, \text{ say.}$$

As in the Case 1,

$$\|(J - h^2 U^*)^{-1} b_1\|_{\infty} \leq \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\} \|W\|,$$

where  $\tilde{W} = [\tilde{W}_1, \tilde{W}_2, \dots, \tilde{W}_{N-1}]^T$ , with  $\tilde{W}_i = d_{i1} + d_{i,N-1}$ ,  $1 \leq i \leq N-1$ .

$$\text{So, } \tilde{W}_i = \frac{\sin(N-i)\theta}{\sin(N\theta)} + \frac{\sin(i\theta)}{\sin(N\theta)}$$

$$= \frac{2 \sin \frac{N\theta}{2} \cdot \cos \left( \frac{N-2i}{2} \theta \right)}{2 \sin \frac{N\theta}{2} \cdot \cos \frac{N\theta}{2}} = \frac{\cos \left( \frac{N-2i}{2} \theta \right)}{\cos \frac{N\theta}{2}} \leq \frac{1}{\cos \frac{N\theta}{2}}, \quad 1 \leq i \leq N-1,$$

so that,

$$(84) \quad \| (J - h^2 U^*)^{-1} b_1 \|_\infty \leq \frac{1}{\cos \frac{N\theta}{2}} \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\}.$$

Again, by (80),

$$\begin{aligned} \| (J - h^2 U^*)^{-1} b_2 \|_\infty &\leq \| (J - h^2 P U^*)^{-1} \|_\infty \| b_2 \|_\infty \\ &\leq \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} h^2 \max_{1 \leq j \leq N-1} |\hat{L}_1 v_j - \hat{L}_1 w_j|, \end{aligned}$$

or,

$$(85) \quad \| (J - h^2 U^*)^{-1} b_2 \|_\infty \leq \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \max_{1 \leq j \leq N-1} |\hat{L}_2 v_j - \hat{L}_2 w_j|.$$

Using (81)-(85) in (75) the result of Case 2 follows.

Stability of Stormer's usual method (9) is discussed in Keller [115], in case when  $-\infty < u^* < 0$ , [Theorem 3.2.1, Keller]. Using the analysis in the proof of Theorem 10, it is possible to give an alternate proof for the stability of Stormer's method both for the cases  $-\infty < u^* < 0$  and  $0 < u^* < \pi^2/(b-a)^2$  as follows:

Proposition 1: Let  $L_1$  be the linear difference operator associated with Stormer's method (9). Then

$$L_1 v_i \equiv - \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} - f(x_i, v_i) = 0; \quad 1 \leq i \leq N-1,$$

If  $f(x, y)$  has continuous derivatives with respect to  $y$ , satisfying

$$u_* \leq \frac{\partial f}{\partial y} \leq u^*,$$

then  $L_1$  is stable in the sense that for all net functions  $\{v_i\}, \{w_i\}$



$$|v_i - w_i| \leq M \left\{ \max \left( |v_0 - w_0|, |v_N - w_N| \right) + \max_{1 \leq j \leq N-1} |L_1 v_j - L_1 w_j| \right\},$$

for  $1 \leq i \leq N-1$ , for some constant  $M$  and some suitable restriction on  $h$  given in the following two cases:

Case 1 :  $-\infty < u^* < 0$ .

Then  $L_1$  is stable for all  $h$  with  $M = M_1$ , where

$$M_1 = \max \left\{ 1, \frac{1}{-u^*} \right\}.$$

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ ,

Then  $L_1$  is stable for all  $h < H_0$ , where  $H_0$  is the smallest positive root of the equation:

$$H^2 u^* - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

with  $M = M_2$ , where  $M_2 = \max \left\{ \frac{1}{\cos \frac{N\theta}{2}}, \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \right\}$ .

Proof: If  $\{v_i\}$  and  $\{w_i\}$  are any two net functions, then as in Theorem 10,

$$h^2 [L_1 v_i - L_1 w_i] = -(v_{i-1} - w_{i-1}) + (2 - h^2 u_i)(v_i - w_i) - (v_{i+1} - w_{i+1}),$$

where  $u_i = \frac{\partial}{\partial y} f(x_i, w_i + \xi_i(v_i - w_i))$ ,  $1 \leq i \leq N-1$ .

In matrix notation, the above system of equations becomes,

$$(86) \quad (J - h^2 U) [v - w] = b_1 + b_2,$$

where  $J, U$  are  $(N-1) \times (N-1)$  matrices given by Definitions 4, 7,

$$[v - w] = [(v_1 - w_1), (v_2 - w_2), \dots, (v_{N-1} - w_{N-1})]^T,$$

$$b_1 = [(v_0 - w_0), 0, \dots, 0, (v_N - w_N)]^T,$$

$$b_2 = h^2 [(L_1 v_1 - L_1 w_1), (L_1 v_2 - L_1 w_2), \dots, (L_1 v_{N-1} - L_1 w_{N-1})]^T.$$

Since  $U \leq U^*$ ,  $(J - h^2 U) \geq (J - h^2 U^*)$ .

If both of these matrices are monotone, then by Theorem 4,

$$(87) \quad (J - h^2 U)^{-1} \leq (J - h^2 U^*)^{-1}.$$

Using (87) in (86), we get

$$[v - w] \leq (J - h^2 U^*)^{-1} (b_1 + b_2).$$

Hence for  $1 \leq i \leq N-1$ ,

$$(88) \quad |v_i - w_i| \leq \| [v - w] \|_\infty \leq \| (J - h^2 U^*)^{-1} b_1 \|_\infty + \| (J - h^2 U^*)^{-1} b_2 \|_\infty.$$

Case 1:  $-\infty < u^* < 0$ .

Now,  $J - h^2 U$  and  $J - h^2 U^*$  are irreducibly diagonally dominant with non-positive off-diagonal elements and positive diagonal elements. Hence by corollary 2, they are monotone matrices. Hence, the inequality (87) holds good. Similar to (79), in this situation

$$(89) \quad \| (J - h^2 U^*)^{-1} b_2 \|_\infty \leq \frac{1}{-u^*} \max_{1 \leq j \leq N-1} |L_1 v_j - L_1 w_j|,$$

and  $\| (J - h^2 U^*)^{-1} b_1 \|_\infty$  is as given in (78). Using (78) and (89) in (88) the result of Case 1 follows.

Case 2:  $0 < u^* < \pi^2 / (b-a)^2$ .

By Lemma 14,  $J - h^2 U$  and  $J - h^2 U^*$  are monotone matrices for all  $h < H_0$ , where  $H_0$  is the smallest positive root of the equation

$$H^2 u^* - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

Hence, the inequality (87) holds good. Similar to (85), we get

$$(90) \quad \| (J - h^2 U^*)^{-1} b_2 \|_{\infty} \leq \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \max_{1 \leq j \leq N-1} |L_2 v_j - L_2 w_j|,$$

and  $\| (J - h^2 U^*)^{-1} b_1 \|_{\infty}$  is as given in (84). Using (84) and (90) in (88), the result of Case 2 follows.

Now as an application of the stability theory of the optimal method we shall establish a relation between the discretization error and the local truncation error for this method.

Let  $\mathcal{L}_h$  be a finite difference operator to solve a BVP (1) of the form

$$(I) \quad \mathcal{L} y(x) \equiv -y'' - f(x, y) = 0, \quad a < x < b,$$

$$y(a) = A, \quad y(b) = B,$$

where  $f(x, y)$  satisfies the conditions stated in (1).

According to Keller [115], let us define the following.

Definition 15: The local truncation errors, in  $\mathcal{L}_h$ , as an approximation to  $\mathcal{L}$ , for any smooth function  $u(x)$ , is defined by

$$\tau_j[u] \equiv \mathcal{L}_h u(x_j) - \mathcal{L} u(x_j), \quad 1 \leq j \leq N-1.$$

Corollary 7: Let  $f(x, y)$  have continuous derivative with respect to  $y$  satisfying  $u_* \leq \frac{\partial f}{\partial y} \leq u^*$ . Then the numerical solution  $\{y_j\}$  of the difference equation

$$\hat{L}_1 v_1 \equiv -\frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} - \hat{\beta}_{1,i} f(x_i, v_i) = 0; \quad 1 \leq i \leq N-1,$$

and the solution  $y(x)$  of the BVP (I) satisfy the relation

$$|Y_j - Y(x_j)| \leq M \max_{1 \leq j \leq N-1} |\hat{\tau}_j^1[Y]|, \quad 0 \leq j \leq N,$$

where  $\hat{\tau}_j^1[u] \equiv \hat{L}_1 u(x_j) - \mathcal{L} u(x_j), \quad 1 \leq j \leq N-1,$

and  $M$  is a constant described in the following two Cases.

Case 1 :  $-\infty < u^* < 0.$

$$M = \hat{M}_1 = \frac{1}{1-\Theta h^2} \max \left\{ 1, \frac{1}{-u^*} \right\}, \quad \text{for all } h \text{ satisfying } \Theta h^2 < 1.$$

Case 2 :  $0 < u^* < \pi^2/(b-a)^2.$

$$M = \hat{M}_2 = \frac{1}{1 - \Theta h^2 u^* \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}} \cdot \max \left\{ \frac{1}{\cos \frac{N\theta}{2}}, \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \right\},$$

for all  $h < \min \{H_1, H_2\}$ ,  $H_1$  is the smallest positive root of the equation:

$$H^2 u^* (1 + \Theta H^2) - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

and  $H_2$  is the smallest positive root of the equation:

$$H^2 u^* - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

provided  $\Theta h^2 < \min \left\{ 1, \frac{8}{(b-a)^2 u^*} \right\}.$

Proof: From (I) and the definition of  $\hat{L}_1$ , evaluated at  $x = x_j$ , we have by the linearity of  $\hat{L}_1$  and the Definition 15,

$$\begin{aligned} \hat{L}_1[Y_j - Y(x_j)] &= \hat{L}_1 Y_j - \hat{L}_1 Y(x_j) = 0 - \hat{L}_1 Y(x_j) \\ &= \mathcal{L} Y(x_j) - \hat{L}_1 Y(x_j) = -\hat{\tau}_j^1[Y]. \end{aligned}$$

Now since,  $Y_0 - Y(x_0) = Y_N - Y(x_N) = 0,$

applying Theorem 10, with  $v_j - w_j = Y_j - Y(x_j)$ , the result follows.

Theorem 11: Let  $\hat{L}_2$  be the linear difference operator associated with the optimal method corresponding to the usual Cowell's method and a net function  $\{v_i\}$  be defined as

$$\hat{L}_2 v_i \equiv - \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2} -$$

$$\left\{ \hat{\beta}_{0,i} f(x_{i-1}, v_{i-1}) + \hat{\beta}_{1,i} f(x_i, v_i) + \hat{\beta}_{2,i} f(x_{i+1}, v_{i+1}) \right\},$$

where  $\hat{\beta}_{k,i}$ ;  $k = 0, 1, 2$ ;  $1 \leq i \leq N-1$ , are the optimal coefficients corresponding to the usual coefficients  $\beta_0 = \frac{1}{12}$ ,  $\beta_1 = \frac{10}{12}$ ,  $\beta_2 = \frac{1}{12}$  and  $\Phi$  be as before, so that  $|\hat{\beta}_{k,i} - \beta_k| \leq \Phi h^2$ ,  $k = 0, 1, 2$ ;  $1 \leq i \leq N-1$ . Assume that  $f(x, y)$  has continuous derivatives with respect to  $y$ ,

satisfying  $u_* \leq \frac{\partial f}{\partial y} \leq u^*$ .

Then  $\hat{L}_2$  is stable in the sense that for all net functions  $\{v_i\}, \{w_i\}$

$$|v_i - w_i| \leq M \left\{ \max \left( |v_0 - w_0|, |v_N - w_N| \right) + \max_{1 \leq j \leq N-1} |\hat{L}_2 v_j - \hat{L}_2 w_j| \right\},$$

$1 \leq i \leq N-1$ , for some constant  $M$  and some suitable restriction on  $h$  given in the following two cases.

Case 1:  $-\infty < u^* < 0$ .

Then  $\hat{L}_2$  is stable with  $M = \hat{M}_3$  where

$$\hat{M}_3 = \frac{1}{1 - \Phi h^2} \max \left\{ \left( 1 + \frac{h^2 u^*}{1 + \frac{h^2 u^*}{12}} \right), \frac{1}{-u^*} \right\}, \quad \text{for all } h < \min \{H_3, H_4\},$$

where  $H_3 \leq \min \left\{ \sqrt{\frac{6}{-u_*}}, \sqrt{\frac{6}{-u^*}} \right\}$ , and  $H_4 \leq \sqrt{\frac{12}{-u^*}}$ , provided  $\Phi h^2 \leq \frac{1}{12}$ .

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

Then  $\hat{L}_2$  is stable with  $M = \hat{M}_4$  where

$$\hat{M}_4 = \frac{1}{1 - \frac{\Phi h^2 u^*}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}}.$$

$$\max \left\{ \left( 1 + \frac{8h^2}{(b-a)^2} \right) \frac{1}{\cos \frac{N\theta}{2}}, \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \right\},$$

for all  $h < \min \{H_5, H_6\}$ ,

provided  $\Phi h^2 \leq \frac{1}{12}$ , where  $H_5 \leq \min \{h_1, h_2\}$ , with  $h_1 = \sqrt{\frac{6}{-u^*}}$ ,  $h_2$  and

$H_6$  are the smallest positive roots of the respective equation:

$$H^2 u^* \left[ (1 + 3\Phi H^2) - \left( \frac{1}{12} + \Phi H^2 \right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0,$$

and

$$h^2 u^* \left( \frac{1}{4 \sin^2 \frac{\pi h}{2(b-a)}} - \frac{1}{12} \right) = 1.$$

Proof: If  $\{v_i\}$  and  $\{w_i\}$  are any two net functions, then

$$h^2 \hat{L}_2 v_i = - (v_{i-1} - 2v_i + v_{i+1}) - h^2 \sum_{k=0}^2 \hat{\beta}_{k,i} f(x_{i+k-1}, v_{i+k-1}),$$

$$h^2 \hat{L}_2 w_i = - (w_{i-1} - 2w_i + w_{i+1}) - h^2 \sum_{k=0}^2 \hat{\beta}_{k,i} f(x_{i+k-1}, w_{i+k-1}).$$

$$\text{So, } h^2 [\hat{L}_2 v_i - \hat{L}_2 w_i] = - (v_{i-1} - w_{i-1}) + 2(v_i - w_i) - (v_{i+1} - w_{i+1})$$

$$- h^2 \sum_{k=0}^2 \hat{\beta}_{k,i} (v_{i+k-1} - w_{i+k-1}) \frac{\partial}{\partial y} f(x_{i+k-1}, w_{i+k-1} + \xi_{i+k-1} (v_{i+k-1} - w_{i+k-1})).$$

Let

$$u_{i+k-1} = \frac{\partial}{\partial y} f(x_{i+k-1}, w_{i+k-1} + \xi_{i+k-1} (v_{i+k-1} - w_{i+k-1})), \quad k=0,1,2; \quad 1 \leq i \leq N-1.$$

Then the above equation becomes

$$h^2 [\hat{L}_2 v_i - \hat{L}_2 w_i] = (-1 - h^2 \hat{\beta}_{0,i} u_{i-1}) (v_{i-1} - w_{i-1}) + (2 - h^2 \hat{\beta}_{1,i} u_i) (v_i - w_i)$$

$$+ (-1-h^2\hat{\beta}_{2,1}u_{1+1})(v_{1+1}-w_{1+1}), \quad \text{for } 1 \leq i \leq N-1.$$

In matrix notation, this system of equations reads

$$(J - h^2\hat{P}U) [v - w] = b_1 + b_2,$$

where  $J$ ,  $\hat{P}$ ,  $U$  are  $(N-1) \times (N-1)$  matrices given by Definitions 4, 6, 7,

$$[v-w] = [(v_1-w_1), (v_2-w_2), \dots, (v_{N-1}-w_{N-1})]^T,$$

$$b_1 = [(1+h^2\hat{\beta}_{0,1}u_0)(v_0-w_0), 0, \dots, 0, (1+h^2\hat{\beta}_{2,N-1}u_N)(v_N-w_N)]^T,$$

$$b_2 = h^2[(\hat{L}_2v_1 - \hat{L}_2w_1), (\hat{L}_2v_2 - \hat{L}_2w_2), \dots, (\hat{L}_2v_{N-1} - \hat{L}_2w_{N-1})]^T.$$

If  $(J - h^2\hat{P}U)$  is a monotone matrix, then it is nonsingular, by Theorem 2. So,

$$(91) \quad [v - w] = (J - h^2\hat{P}U)^{-1}(b_1 + b_2).$$

Now  $u_* \leq u \leq u^*$ , so that  $U_* \leq U \leq U^*$ ,

where  $U$ ,  $U_*$ ,  $U^*$  are the matrices given by Definitions 7, 8, 9.

Hence,

$$(J - h^2\hat{P}U) \geq (J - h^2\hat{P}U^*).$$

If both of these matrices are monotone, then by Theorem 4,

$$(92) \quad (J - h^2\hat{P}U)^{-1} \leq (J - h^2\hat{P}U^*)^{-1}.$$

Using (92) in (91), we get

$$(93) \quad [v - w] \leq (J - h^2\hat{P}U^*)^{-1}(b_1 + b_2).$$

According to Lemma 4, we can write,

$$\hat{\beta}_{1n} - \beta_1 = \phi_{1n} h^2, \quad i = 0, 1, 2; \quad 1 \leq n \leq N-1.$$

Let  $\hat{\phi} = (\hat{\phi}_{ij})_{i,j=1}^{N-1}$  be a tridiagonal matrix, where

$$\hat{\phi}_{11} = \phi_{1,1}, \quad 1 \leq i \leq N-1; \quad \hat{\phi}_{1,1-1} = \phi_{0,1}, \quad 2 \leq i \leq N-1; \quad \hat{\phi}_{1,1+1} = \phi_{2,1}, \quad 1 \leq i \leq N-2.$$

Then,

$$(J - h^2 \hat{P}U^*) = J - h^2(P + h^2 \hat{\phi})U^* = (J - h^2 PU^*) - h^4 \hat{\phi}U^*,$$

where  $P$  is a matrix given by Definition 10. If

$$(94) \quad \|(J - h^2 PU^*)^{-1} h^4 \hat{\phi}U^*\|_{\infty} \leq k < 1,$$

then by Lemma 2,  $[I - (J - h^2 PU^*)^{-1} h^4 \hat{\phi}U^*]^{-1}$  exists and

$$\|[I - (J - h^2 PU^*)^{-1} h^4 \hat{\phi}U^*]^{-1}\|_{\infty} \leq \frac{1}{1-k}.$$

$$\text{So,} \quad (J - h^2 \hat{P}U^*)^{-1} = [I - (J - h^2 PU^*)^{-1} h^4 \hat{\phi}U^*]^{-1} (J - h^2 PU^*)^{-1}.$$

Hence from (93) we get,

$$[v - w] \leq [I - (J - h^2 PU^*)^{-1} h^4 \hat{\phi}U^*]^{-1} (J - h^2 PU^*)^{-1} (b_1 + b_2),$$

and for  $1 \leq i \leq N-1$ ,

$$(95) \quad |v_i - w_i| \leq \| [v - w] \|_{\infty} \leq \| [I - (J - h^2 PU^*)^{-1} h^4 \hat{\phi}U^*]^{-1} \|_{\infty} \cdot \left( \| (J - h^2 PU^*)^{-1} b_1 \|_{\infty} + \| (J - h^2 PU^*)^{-1} b_2 \|_{\infty} \right).$$

$$\text{Let } J - h^2 PU^* = (\hat{g}_{nm})_{n,m=1}^{N-1}, \quad \text{and } (J - h^2 PU^*)^{-1} = (g_{nm})_{n,m=1}^{N-1}.$$

Case 1:  $-\infty < u^* < 0$ .

By Lemma 8 and Lemma 12,  $J - h^2 \hat{P}U$  and  $J - h^2 \hat{P}U^*$  are monotone matrices for all  $h < H_3$ , provided  $\Phi h^2 \leq \frac{1}{12}$ ,

$$\text{where} \quad H_3 \leq \min \left\{ \sqrt{\frac{6}{-u_*}}, \sqrt{\frac{6}{-u^*}} \right\}.$$

Hence for these  $h$ , the relations (91) and (92) hold good. Again by Lemma 16,  $J - h^2 PU^*$  is monotone for all  $h < H_4$ ,



where

$$H_4 = \sqrt{\frac{12}{-u^*}}.$$

Then by Theorem 2,  $(J - h^2 P U^*)^{-1} \geq 0$  i.e.,  $g_{nm} \geq 0$ . By Lemma 18, we have

$$(J - h^2 P U^*)^{-1}_{n,m} = g_{nm} = \begin{cases} \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sinh(n\theta) \cdot \sinh(N-m)\theta}{\sinh(N\theta) \cdot \sinh\theta}; & n \leq m \\ \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sinh(m\theta) \cdot \sinh(N-n)\theta}{\sinh(N\theta) \cdot \sinh\theta}; & n \geq m, \end{cases}$$

$$\text{where } \sinh(\theta/2) = \frac{h}{2} \sqrt{\frac{-u^*}{1 + \frac{h^2 u^*}{12}}}.$$

Now we shall calculate  $\|(J - h^2 P U^*)^{-1}\|_\infty$ . Let  $a_m$  denote the sum of the elements in the  $m$ -th row of the matrix  $J - h^2 P U^*$ . Then as in the Case 1 of Theorem 10,

$$\sum_{m=1}^{N-1} g_{nm} a_m = 1; \quad \text{for } 1 \leq n \leq N-1.$$

$$\text{Now, } a_1 = a_{N-1} = 1 - \frac{11}{12} h^2 u^* \geq -h^2 u^*, \quad \text{if } h \leq \sqrt{\frac{12}{-u^*}}.$$

$$\text{Also, } a_m = -h^2 u^*, \quad \text{for } 2 \leq m \leq N-2.$$

$$\text{So, } a_m \geq -h^2 u^*, \quad \text{for } 1 \leq m \leq N-1, \quad \text{if } h \leq \sqrt{\frac{12}{-u^*}}.$$

$$\text{So, } -h^2 u^* \sum_{m=1}^{N-1} g_{nm} \leq \sum_{m=1}^{N-1} g_{nm} a_m = 1, \quad \text{for } 1 \leq n \leq N-1,$$

$$\text{and } \sum_{m=1}^{N-1} g_{nm} \leq \frac{1}{-h^2 u^*}, \quad \text{for } 1 \leq n \leq N-1.$$

$$\text{So,} \quad \| (J - h^2 P U^*)^{-1} \|_{\infty} = \max_{1 \leq n \leq N-1} \sum_{m=1}^{N-1} g_{nm} \leq \frac{1}{-h^2 u^*}.$$

Hence, using Corollary 3, we have

$$\begin{aligned} \| (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^* \|_{\infty} &\leq \| (J - h^2 P U^*)^{-1} \|_{\infty} h^4 \| \hat{\phi} \|_{\infty} \| U^* \|_{\infty} \\ &\leq \frac{1}{-h^2 u^*} h^4 \Phi (-u^*) = \Phi h^2. \end{aligned}$$

Thus as stated in (94), if  $\Phi h^2 < 1$ , then the matrix

$$[I - (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*]^{-1} \text{ exists and}$$

$$(96) \quad \| [I - (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*]^{-1} \|_{\infty} \leq \frac{1}{1 - \Phi h^2} = c_3, \text{ say.}$$

$$\text{Let } (J - h^2 P U^*)^{-1} b_1 = [\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_{N-1}]^T, \text{ where}$$

$$\tilde{v}_i = (1 + h^2 \hat{\beta}_{0,1} u_0) (v_0 - w_0) g_{i1} + (1 + h^2 \hat{\beta}_{2,N-1} u_N) (v_N - w_N) g_{i,N-1}, \quad 1 \leq i \leq N-1,$$

$$\text{and } \tilde{w} = [\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_{N-1}]^T,$$

$$\text{with } \tilde{w}_i = g_{i1} + g_{i,N-1}, \quad 1 \leq i \leq N-1.$$

Then

$$\| (J - h^2 P U^*)^{-1} b_1 \|_{\infty} \leq \left\{ 1 + h^2 \left( \frac{1}{12} + \Phi h^2 \right) u^* \right\} \cdot \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\} \| \tilde{w} \|_{\infty}.$$

Now as in the Case 1 of Theorem 10, for  $1 \leq i \leq N-1$

$$\tilde{w}_i = g_{i1} + g_{i,N-1} = \frac{1}{1 + \frac{h^2 u^*}{12}} \left( \frac{\sinh((N-i)\theta)}{\sinh(N\theta)} + \frac{\sinh(i\theta)}{\sinh(N\theta)} \right) \leq \frac{1}{1 + \frac{h^2 u^*}{12}}$$

$$\text{So, } \| \tilde{w} \|_{\infty} \leq \frac{1}{1 + \frac{h^2 u^*}{12}}. \text{ Using } \Phi h^2 < 1,$$

$$(97) \quad \| (J - h^2 P U^*)^{-1} b_1 \|_{\infty} \leq \frac{\left\{ 1 + h^2 \left( \frac{1}{12} + \Phi h^2 \right) u^* \right\}}{1 + \frac{h^2 u^*}{12}} \cdot \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\}$$

$$< \left( 1 + \frac{h^2 u^*}{1 + \frac{h^2 u^*}{12}} \right) \cdot \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\}.$$

Again,  $\| (J - h^2 P U^*)^{-1} b_2 \|_\infty \leq \| (J - h^2 P U^*)^{-1} \|_\infty \| b_2 \|_\infty$

$$\leq \frac{1}{-h^2 u^*} h^2 \max_{1 \leq j \leq N-1} | \hat{L}_2 v_j - \hat{L}_2 w_j |.$$

So,

$$(98) \quad \| (J - h^2 P U^*)^{-1} b_2 \|_\infty \leq \frac{1}{-u^*} \max_{1 \leq j \leq N-1} | \hat{L}_2 v_j - \hat{L}_2 w_j |.$$

Using (96), (97) and (98) in (95) the result of Case 1 follows.

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

By Lemma 9 and Lemma 13,  $J - h^2 \hat{P} U$  and  $J - h^2 \hat{P} U^*$  are monotone for all  $h < H_5$ , provided  $\Phi h^2 \leq \frac{1}{12}$ , where  $H_5 \leq \min\{h_1, h_2\}$ , with  $h_1 = \sqrt{\frac{6}{-u^*}}$  and  $h_2$  is the smallest positive root of the equation :

$$h^2 u^* \left[ (1 + 3\Phi H^2) - \left( \frac{1}{12} + \Phi H^2 \right) 4 \sin^2 \frac{\pi H}{2(b-a)} \right] - 4 \sin^2 \frac{\pi H}{2(b-a)} = 0.$$

Hence for these  $h$ , the relations (91) and (92) hold good. Again by Lemma 17,  $J - h^2 P U^*$  is monotone for all  $h < H_6$ , where  $H_6$  is the root of the equation,

$$h^2 u^* \left( \frac{1}{4 \sin^2 \frac{\pi h}{2(b-a)}} - \frac{1}{12} \right) = 1.$$

So, by Theorem 2,  $(J - h^2 P U^*)^{-1} \geq \underline{0}$ , i.e.,  $g_{nm} \geq 0$ .

By Lemma 19, we get

$$(J - h^2 P U^*)^{-1}_{n,m} = g_{nm} = \begin{cases} \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sin(n\theta) \cdot \sin(N-m)\theta}{\sin(N\theta) \cdot \sin\theta}; & n \leq m \\ \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{\sin(m\theta) \cdot \sin(N-n)\theta}{\sin(N\theta) \cdot \sin\theta}; & n \geq m, \end{cases}$$

$$\text{where } \sin(\theta/2) = \frac{h}{2} \sqrt{\frac{u^*}{1 + \frac{h^2 u^*}{12}}}.$$

Now we shall calculate  $\|(J - h^2 P U^*)^{-1}\|_{\infty}$ . As in the Case 2 of Theorem 10, we have

$$\begin{aligned} \sum_{m=1}^{N-1} |g_{nm}| &= \sum_{m=1}^n g_{nm} + \sum_{m=n+1}^{N-1} g_{nm} \\ &= \frac{1}{1 + \frac{h^2 u^*}{12}} \left[ \sum_{m=1}^n \frac{\sin(m\theta) \cdot \sin(N-n)\theta}{\sin(N\theta) \cdot \sin\theta} + \sum_{m=n+1}^{N-1} \frac{\sin(n\theta) \cdot \sin(N-m)\theta}{\sin(N\theta) \cdot \sin\theta} \right] \\ &\leq \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}, \end{aligned}$$

and

$$(99) \quad \|(J - h^2 P U^*)^{-1}\|_{\infty} \leq \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta}.$$

$$\text{Hence, } \|(J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*\|_{\infty} \leq \|(J - h^2 P U^*)^{-1}\|_{\infty} h^4 \|\hat{\phi}\|_{\infty} \|U^*\|_{\infty},$$

so that

$$(100) \quad \|(J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*\|_{\infty} \leq \frac{\Phi h^2 u^*}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} = k, \text{ say.}$$

Hence as stated in (94), if  $k < 1$ , i.e., if

$$(101) \quad \Phi h^2 u^* < \left(1 + \frac{h^2 u^*}{12}\right) \frac{8}{(b-a)^2},$$

then the matrix  $[I - (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*]^{-1}$  exists and

$$(102) \quad \| [I - (J - h^2 P U^*)^{-1} h^4 \hat{\phi} U^*]^{-1} \|_{\infty} \leq \frac{1}{1-k} = C_4, \text{ say.}$$

As in the Case 1,

$$\| (J - h^2 P U^*)^{-1} b_1 \|_{\infty} \leq \left\{ 1 + h^2 \left( \frac{1}{12} + \Phi h^2 \right) u^* \right\} \cdot \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\} \| \tilde{W} \|,$$

where  $\tilde{W} = [\tilde{W}_1, \tilde{W}_2, \dots, \tilde{W}_{N-1}]^T$ , with

$$\begin{aligned} \tilde{W}_i &= g_{i,1} + g_{i,N-1}, \\ &= \frac{1}{1 + \frac{h^2 u^*}{12}} \left( \frac{\sin(N-i)\theta}{\sin(N\theta)} + \frac{\sin(i\theta)}{\sin(N\theta)} \right) \leq \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{1}{\cos \frac{N\theta}{2}}, \quad 1 \leq i \leq N-1. \end{aligned}$$

As in the Case2 of Theorem 10, using (101),

$$\begin{aligned} (103) \quad \| (J - h^2 P U^*)^{-1} b_1 \|_{\infty} &\leq \frac{\left\{ 1 + h^2 \left( \frac{1}{12} + \Phi h^2 \right) u^* \right\}}{1 + \frac{h^2 u^*}{12}} \frac{1}{\cos \frac{N\theta}{2}} \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\} \\ &\leq \left( 1 + \frac{8h^2}{(b-a)^2} \right) \frac{1}{\cos \frac{N\theta}{2}} \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\}. \end{aligned}$$

Again, by (99),

$$\begin{aligned} \| (J - h^2 P U^*)^{-1} b_2 \|_{\infty} &\leq \| (J - h^2 P U^*)^{-1} \|_{\infty} \| b_2 \|_{\infty} \\ &\leq \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{N^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} h^2 \max_{1 \leq j \leq N-1} | \hat{L}_2 v_j - \hat{L}_2 w_j |, \end{aligned}$$

so that

$$(104) \quad \| (J - h^2 P U^*)^{-1} b_2 \|_{\infty} \leq \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \max_{1 \leq j \leq N-1} | \hat{L}_2 v_j - \hat{L}_2 w_j |.$$

Using (100)-(104) in (95) the result of Case 2 follows.

Now we shall prove the stability of Cowell's usual method (5) with function evaluation at three points, along the lines of the proof of Theorem 11.

Proposition 2: Let  $L_2$  be the linear difference operator associated with Cowell's method. With the net function  $\{v_i\}$  defined as

$$L_2 v_i \equiv - \frac{v_{i-1} - 2v_i + v_{i+1}}{h^2}$$

$$\left\{ \frac{1}{12} f(x_{i-1}, v_{i-1}) + \frac{10}{12} f(x_i, v_i) + \frac{1}{12} f(x_{i+1}, v_{i+1}) \right\},$$

assume that  $f(x, y)$  has continuous derivatives with respect to  $y$ ,

satisfying 
$$u_* \leq \frac{\partial f}{\partial y} \leq u^*.$$

Then  $L_2$  is stable in the sense that for all net functions  $\{v_i\}, \{w_i\}$

$$|v_i - w_i| \leq M \left\{ \max \left( |v_0 - w_0|, |v_N - w_N| \right) + \max_{1 \leq j \leq N-1} |L_2 v_j - L_2 w_j| \right\},$$

$1 \leq i \leq N-1$ , for some constant  $M$  and some suitable restriction on  $h$  given in the following two cases.

Case 1 :  $-\infty < u^* < 0$ .

Then  $L_2$  is stable for all  $h < H_4$ , with  $M = M_3 = \max \left\{ 1, \frac{1}{-u_*} \right\}$ ,

for all  $h < H_4$ , where  $H_4 \leq \sqrt{\frac{12}{-u_*}}$ .

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

Then  $L_2$  is stable for all  $h < \min \{H_6, H_7\}$ ,

where  $H_7 = \sqrt{\frac{12}{-\bar{u}_*}}$ ,  $\bar{u}_* = \min\{0, u_*\}$ , and  $H_6$  is the smallest positive

root of the equations:

$$h^2 u^* \left( \frac{1}{4 \sin^2 \frac{\pi h}{2(b-a)}} - \frac{1}{12} \right) = 1,$$

with  $M = M_4$ , where

$$M_4 = \max \left\{ \frac{1}{\cos \frac{N\theta}{2}}, \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \right\}.$$

Proof: If  $\{v_i\}$  and  $\{w_i\}$  are any two net functions, then setting

$$\beta_0 = \frac{1}{12}, \quad \beta_1 = \frac{10}{12}, \quad \beta_2 = \frac{1}{12} \quad \text{and}$$

$$u_{i+k-1} = \frac{\partial}{\partial y} f(x_{i+k-1}, w_{i+k-1} + \xi_{i+k-1} (v_{i+k-1} - w_{i+k-1})), \quad k=0,1,2; \quad 1 \leq i \leq N-1,$$

as in Theorem 11, we get

$$\begin{aligned} h^2 [L_2 v_i - L_2 w_i] &= (-1 - \frac{h^2}{12} u_{i-1}) (v_{i-1} - w_{i-1}) + (2 - \frac{10}{12} h^2 u_i) (v_i - w_i) \\ &\quad + (-1 - \frac{h^2}{12} u_{i+1}) (v_{i+1} - w_{i+1}), \quad 1 \leq i \leq N-1. \end{aligned}$$

In matrix notation, this system of equations reads

$$(J - h^2 PU) [v-w] = b_1 + b_2,$$

where  $J$ ,  $P$ ,  $U$  are  $(N-1) \times (N-1)$  matrices given by Definitions 4, 10, 7;

$$[v-w] = [(v_1 - w_1), (v_2 - w_2), \dots, (v_{N-1} - w_{N-1})]^T,$$

$$b_1 = [(1 + \frac{h^2}{12} u_0) (v_0 - w_0), 0, \dots, 0, (1 + \frac{h^2}{12} u_N) (v_N - w_N)]^T \quad \text{and}$$

$$b_2 = h^2 [(L_2 v_1 - L_2 w_1), (L_2 v_2 - L_2 w_2), \dots, (L_2 v_{N-1} - L_2 w_{N-1})]^T.$$

If  $(J - h^2 PU)$  is a nonsingular matrix, then

$$(105) \quad [v - w] = (J - h^2 PU)^{-1} (b_1 + b_2).$$

Since  $U \leq U^*$ ,  $(J - h^2 PU) \geq (J - h^2 PU^*)$ .

If both of these matrices are monotone then by Theorem 4,

$$(106) \quad (J - h^2 PU)^{-1} \leq (J - h^2 PU^*)^{-1}.$$

Using (106) in (105) and taking norm, we get

$$(107) \quad |v_i - w_i| \leq \| [v - w] \|_\infty \leq \| (J - h^2 PU^*)^{-1} b_1 \|_\infty + \| (J - h^2 PU^*)^{-1} b_2 \|_\infty, \\ 1 \leq i \leq N-1.$$

Case 1 :  $-\infty < u^* < 0$ .

By Lemmas 15 and 16,  $J - h^2 PU$  and  $J - h^2 PU^*$  are monotone for all  $h < H_4$ , where  $H_4 \leq \min \left\{ \sqrt{\frac{12}{-u_*}}, \sqrt{\frac{12}{-u^*}} \right\} = \sqrt{\frac{12}{-u_*}}$ . Hence for all  $h < H_4$ , the relations (105) and (106) hold good. As in Case 1 of Theorem 11,

$$\| (J - h^2 PU^*)^{-1} b_1 \|_\infty \leq (1 + \frac{h^2 u^*}{12}) \cdot \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\} \| \tilde{W} \|_\infty,$$

where  $\tilde{W}$  is as defined in the proof of Case 1 of Theorem 11, with

$$\| \tilde{W} \|_\infty \leq \frac{1}{(1 + \frac{h^2 u^*}{12})},$$

$$(108) \quad \| (J - h^2 PU^*)^{-1} b_1 \|_\infty \leq \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\}.$$

Similar to (98), we get

$$(109) \quad \| (J - h^2 PU^*)^{-1} b_2 \|_\infty \leq \frac{1}{-u^*} \max_{1 \leq j \leq N-1} |L_2 v_j - L_2 w_j|.$$

Using (108) and (109) in (107) the result of Case 1 follows.

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

Following the proof of Lemma 9 (essentially replacing  $\Phi$  by 0), we easily obtain that  $J - h^2 PU$  is monotone for all  $h < \min \{H_6, H_7\}$ , where  $H_7$  where  $H_7 = \sqrt{\frac{12}{-\bar{u}_*}}$ ,  $\bar{u}_*$  is smallest of 0 and  $u_*$ , and  $H_6$  is the

smallest positive root of the equation :



$$h^2 u^* \left( \frac{1}{4 \sin^2 \frac{\pi h}{2(b-a)}} - \frac{1}{12} \right) = 1.$$

By Lemma 17,  $J - h^2 P U^*$  is monotone for all  $h < H_6$ , Hence for all  $h < \min\{H_6, H_7\}$ , the relations (105) and (106) hold good. As in Case 2 of Theorem 11, we get

$$(110) \quad \|(J - h^2 P U^*)^{-1} b_1\|_{\infty} \leq \frac{1}{\cos \frac{N\theta}{2}} \max \left\{ |v_0 - w_0|, |v_N - w_N| \right\}.$$

Similar to (104),

$$(111) \quad \|(J - h^2 P U^*)^{-1} b_2\|_{\infty} \leq \frac{1}{1 + \frac{h^2 u^*}{12}} \frac{(b-a)^2}{8} \frac{N\theta}{\sin(N\theta)} \frac{\theta}{\sin\theta} \max_{1 \leq j \leq N-1} |L_2 v_j - L_2 w_j|.$$

Using (110) and (111) in (107) the result of Case 2 follows.

Now similar to Corollary 7, we shall establish the relation between the discretisation error and the local truncation error for the optimal and the usual method with function evaluation at three points.

Corollary 8: Let  $f(x, y)$  have continuous derivatives with respect to  $y$  which satisfy  $u_* \leq \frac{\partial f}{\partial y} \leq u^*$ . Then the numerical solution  $\{y_j\}$  of the difference equation

$$\hat{L}_2 v_1 \equiv - \frac{v_{1-1} - 2v_1 + v_{1+1}}{h^2} -$$

$$\left\{ \hat{\beta}_{0,1} f(x_{1-1}, v_{1-1}) + \hat{\beta}_{1,1} f(x_1, v_1) + \hat{\beta}_{2,1} f(x_{1+1}, v_{1+1}) \right\},$$

and the solution  $y(x)$  of the BVP (I) satisfy,

$$|y_j - y(x_j)| \leq M \max_{1 \leq j \leq N-1} |\hat{\tau}_j^2[y]|, \quad 0 \leq j \leq N,$$

where  $\hat{\tau}_j^2[u] \equiv \hat{L}_2 u(x_j) - \mathcal{L} u(x_j), \quad 1 \leq j \leq N-1,$

Case 1 :  $-\infty < u^* < 0$ .

$M = \hat{M}_3$ , for all  $h < \min\{H_3, H_4\}$ , provided  $\phi h^2 \leq \frac{1}{12}$ .

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

$M = \hat{M}_4$ , for all  $h < \min\{H_5, H_6\}$ ,

where  $\hat{M}_3, \hat{M}_4, H_3, H_4, H_5, H_6$  are as given in Theorem 11.

Proof: From (I) and the definition of  $\hat{L}_2$ , evaluated at  $x = x_j$ , by the linearity of  $\hat{L}_2$  and the Definition 15, as in Corollary 7, we get

$$\hat{L}_2[y_j - y(x_j)] = -\hat{\tau}_j^2[y].$$

Now since,  $y_0 - y(x_0) = y_N - y(x_N) = 0$ , applying Theorem 11, with  $v_j - w_j = y_j - y(x_j)$ , the result follows.

Corollary 9: Let  $f(x,y)$  have continuous derivatives with respect to  $y$  which satisfy  $u_* \leq \frac{\partial f}{\partial y} \leq u^*$ . Then the numerical solution  $\{y_j\}$  of the difference equation

$$L_2 v_1 \equiv -\frac{v_{1-1} - 2v_1 + v_{1+1}}{h^2} -$$

$$\left\{ \frac{1}{12} f(x_{1-1}, v_{1-1}) + \frac{10}{12} f(x_1, v_1) + \frac{1}{12} f(x_{1+1}, v_{1+1}) \right\}$$

and the solution  $y(x)$  of the BVP (I) satisfy,

$$|y_j - y(x_j)| \leq M \max_{1 \leq j \leq N-1} |\tau_j^2[y]|, \quad 0 \leq j \leq N,$$

where  $\tau_j^2[u] \equiv L_2 u(x_j) - \mathcal{L} u(x_j)$ ,  $1 \leq j \leq N-1$ .

Case1 :  $-\infty < u^* < 0$ .

$M = M_3$ , for all  $h < H_4$ .

Case 2:  $0 < u^* < \pi^2/(b-a)^2$ .

with  $M = M_4$ , for all  $h < \min \{H_6, H_7\}$ ,

where  $M_3, M_4, H_4, H_6, H_7$  are as given in Proposition 2.

Proof: From (I) and the definition of  $L_2$ , evaluated at  $x = x_j$ , by the linearity of  $L_2$  and the Definition 15, as in Corollary 7 we get

$$L_2[y_j - y(x_j)] = -\tau_j^2 [y].$$

Now since,  $y_0 - y(x_0) = y_N - y(x_N) = 0$ , applying Theorem 11 with

$v_j - w_j = y_j - y(x_j)$ , the result follows.

## CHAPTER - 4

### OPTIMAL MULTISTEP METHODS IN $H^2(C_r)$ SPACE

#### 4.1 Introduction

Quadrature formulae of various types with functions in space  $H^2(C_r)$  have been discussed by Larkin [126]; Ritcher [157], [158], [159]; Chawla and Kaul [36], [37], [38]; Finney and Price Jr. [82]; Brij Bhusan [27] and others. Optimal multistep methods in this space for first order differential equations have been studied by Brij Bhusan [27]. In this chapter we shall study optimal multistep methods of various kinds discussed in Chapter 1, in the space  $H^2(C_r)$ , for the second order differential equations.

For the purpose of numerical illustration, as usual methods, we have taken Cowell's method with function evaluation at three points for both initial and boundary value problems, Stormer's method with function evaluation at five points in case of initial value problem only and Stormer's method with function evaluation at one point in case of boundary value problem only. We observe that the formulae obtained in this chapter are better suited for differential equations whose solutions have more pronounced singularities outside  $C_r$  but not too far from the boundary of  $C_r$ .

The norm in the space  $H^2(C_r)$  is based on the line integral along the contour  $C_r$ . Hence the points in the neighbourhood of  $C_r$  get more weightage than the points inside  $C_r$  for the optimal multistep methods obtained in the space  $H^2(C_r)$ . So, the optimal methods in  $H^2(C_r)$  space are more effective in the peripheral region.

of the subinterval  $[a,b]$  of  $(-r,r)$ .

In section 4.2, we prove that the second derivative of the kernel function of  $H^2(C_r)$  space is the complex conjugate of the representer for the second derivative evaluation functional in  $H^2(C_r)$ . We devote section 4.3, for a discussion of optimal multistep methods in  $H^2(C_r)$  and their numerical implementation; section 4.4, for a discussion of optimal multistep methods interpolatory for polynomials of certain degree and their numerical illustrations; section 4.5, for a discussion of optimal multistep methods interpolatory for certain other functions and their numerical illustrations. In section 4.6, we describe the limiting behaviour of the optimal coefficients as  $r \rightarrow \infty$ . In section 4.7, we are calculating and comparing the asymptotic behaviour of the local truncation error (LTE) functionals for usual and the  $\beta$ -optimal methods in  $H^2(C_r)$  space.

## 4.2 The Hilbert space $H^2(C_r)$

The space  $H^2(C_r)$  consists of functions of a complex variable  $z$  which are analytic within the disc  $D_r = \{z : |z| < r\}$  and square integrable on the circle  $C_r = \{z : |z| = r\}$ . The space  $H^2(C_r)$  is a Hilbert space with the inner product defined by

$$(1) \quad (f, g) = \int_{C_r} f(z) \overline{g(z)} ds, \quad f, g \in H^2(C_r),$$

the integration being with respect to the length element  $ds$  on the circle.

This space possesses a reproducing kernel function given by

$$(2) \quad K(z, \bar{t}) = \frac{r}{2\pi(r^2 - z\bar{t})}$$

and a complete orthonormal sequence  $\{\psi_k\}$  of functions (Szegő [177])

$$(3) \quad \psi_k(z) = \frac{z^k}{\sqrt{2\pi r} r^k}, \quad k = 0, 1, 2, \dots$$

As the space  $H^2(C_r)$  possesses a reproducing kernel function, the linear functional for point evaluation

$$L_z : f \longrightarrow f(z), \quad f \in H^2(C_r), \quad z \in D_r$$

is a bounded linear functional (Aronszajn [4]).

Also the representer of  $k$ -th derivative evaluation functional

$$Dk(z) : f \longrightarrow f^k(z); \quad f \in H^2(C_r), \quad z \in D_r,$$

where  $k$  is a positive integer, is a bounded linear functional and with a bound given as follows.

Theorem 1: (Brij Bhusan [27]) For  $z \in D_r$  and  $k$  any positive integer,  $Dk(z)$  is a bounded linear functional in  $H^2(C_r)$  with

$$\| D_z^k \| \leq \frac{k! \sqrt{r}}{\sqrt{2\pi}(r - |z|)^{k+1}}.$$

The representer  $D2(t, \bar{z}_0)$  of the second derivative evaluation functional at a point  $z_0 \in D_r$ , should satisfy

$$D2(t, \bar{z}_0) = \overline{\frac{\partial^2}{\partial z^2} K(z, \bar{t})} \Big|_{z=z_0} = \frac{rt^2}{\pi(r^2 - \bar{z}_0 t)^3}.$$

Theorem 2: The function

$$D2(t, \bar{z}_0) = \frac{rt^2}{\pi(r^2 - \bar{z}_0 t)^3}$$

is the representer for the 2-nd derivative evaluation functional at  $z_0 \in D_r$ .

Proof: The only singularity of  $D_2(t, \bar{z}_0)$  is a pole of order 3 at  $t = \frac{r^2}{\bar{z}_0}$ . As  $z_0 \in D_r$ ,  $\frac{r^2}{\bar{z}_0}$  lies outside  $C_r$ . Hence the function  $D_2(t, \bar{z}_0)$  belongs to  $H^2(C_r)$ . Now we have to show that

$$f''(z_0) = \left( f(t), \frac{rt^2}{\pi(r^2 - \bar{z}_0 t)^3} \right).$$

By Cauchy integral formula

$$\begin{aligned} f''(z_0) &= \frac{2!}{2\pi i} \int_{C_r} \frac{f(t)}{(t-z_0)^3} dt \\ &= \frac{1}{\pi i} \int_0^{2\pi} \frac{f(t)}{(re^{i\theta} - z_0)^3} re^{i\theta} i d\theta, \quad \text{putting } t = re^{i\theta} \\ &= \frac{r}{\pi} \int_0^{2\pi} \frac{f(t) r^2 e^{-2i\theta}}{(re^{i\theta} - z_0)^3 r^3 e^{-3i\theta}} r d\theta \\ &= \int_{C_r} f(t) \left\{ \frac{r}{\pi} \frac{(re^{i\theta})^2}{(r^2 - \bar{z}_0 re^{i\theta})^3} \right\} ds \\ &= \left( f(t), \frac{r}{\pi} \frac{t^2}{(r^2 - \bar{z}_0 t)^3} \right), \end{aligned}$$

which proves the result.

In  $H^2(C_r)$  space we have the following :

$$(4) \quad K(z, \bar{t}) = \frac{r}{2\pi(r^2 - z\bar{t})}$$

$$(5) \quad D_2(t, \bar{z}) = \frac{\partial^2}{\partial z^2} K(z, \bar{t}) = \frac{rt^2}{\pi(r^2 - t\bar{z})^3}$$

$$(6) \quad D_2''(t, \bar{z}) = \frac{\partial^2}{\partial t^2} D_2(t, \bar{z}) = \frac{r}{\pi} \frac{2}{(r^2 - t\bar{z})^3} \left[ 1 + \frac{6t\bar{z}}{(r^2 - t\bar{z})} + \frac{6t^2\bar{z}^2}{(r^2 - t\bar{z})^2} \right]$$

Now we shall implement in  $H^2(C_r)$  space, (i)  $\beta$ -optimal method, (ii)  $\alpha$ -optimal method, (iii)  $\beta$ -optimal method with restriction, (iv)  $\beta$ -optimal method interpolatory for polynomials, and (v)

$\beta$ -optimal method interpolatory for linearly independent functions  $\exp(1.6x)$  and  $\exp(-1.6x)$ , on twenty four BVP's and twenty four IVP's with differential equations listed below, using Cowell's method with function evaluation at three points as the usual method in which  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=1/12$ ,  $\beta_1=10/12$ , and  $\beta_2=1/12$ . We also implement (vi)  $\beta$ -optimal method, on twenty four BVP's using Stormer's method with function evaluation at one point as the usual method in which  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_1=1$ , and (vii)  $\beta$ -optimal method, on twenty four IVP's using Stormer's explicit method with function evaluation at five points as the usual method in which  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=299/240$ ,  $\beta_1=-11/15$ ,  $\beta_2=97/120$ ,  $\beta_3=-2/5$ , and  $\beta_4=19/240$ . The initial values for the IVP's and the boundary values for the BVP's are calculated from the analytical solutions. Tables for optimal beta coefficients are common for BVP-s and IVP-s if the differential equation is written in the common form  $y'' = f(x,y)$ . Throughout this thesis, the numerical experiments are done with this convention. The Key symbols used in the following tables are as follows:

r-sum : row sum of  $\hat{\beta}_{i,n}$ 's,

$\|T_n\|^2$  : square norm of LTE functional for usual method,

$\|\hat{T}_n\|^2$  : square norm of LTE functional for optimal method,

itu : number of iterations for convergence of the solution using usual method,

ito : number of iterations for convergence of the solution using optimal method,

$\bar{e}_u$  : average of absolute values of point wise discretisation errors for usual method,



$\bar{e}_0$  : average of absolute values of point wise discretisation errors for optimal method.

The twenty four differential equations and their analytical (particular) solutions are listed below.

1.  $\frac{d^2 y}{dx^2} = y^3(1 + 3x^2 y^2); \quad y = (2.01^2 - x^2)^{-1/2}.$
2.  $\frac{d^2 y}{dx^2} = y^3(1 + 3x^2 y^2); \quad y = (2.11^2 - x^2)^{-1/2}.$
3.  $\frac{d^2 y}{dx^2} = 2y^2(1 + 4x^2 y); \quad y = (2.03^2 - x^2)^{-1}.$
4.  $\frac{d^2 y}{dx^2} = 2y^2(1 + 4x^2 y); \quad y = (2.11^2 - x^2)^{-1}.$
5.  $\frac{d^2 y}{dx^2} = -2e^{-y}(1 + 2x^2 e^{-y}); \quad y = \log(2.01^2 - x^2).$
6.  $\frac{d^2 y}{dx^2} = 4x^2 y^2(3 + 8x^4 y); \quad y = (2.01^4 - x^4)^{-1}.$
7.  $\frac{d^2 y}{dx^2} = 6x^4 y^2(5 + 12x^6 y); \quad y = (2.01^6 - x^6)^{-1}.$
8.  $\frac{d^2 y}{dx^2} = 6x^2 y^3(1 + 2x^4 y^2); \quad y = (2.01^4 - x^4)^{-1/2}.$
9.  $\frac{d^2 y}{dx^2} = 3x^4 y^3(5 + 9x^6 y^2); \quad y = (2.01^6 - x^6)^{-1/2}.$
10.  $\frac{d^2 y}{dx^2} = 2y(1 + 2x^2); \quad y = e^{x^2}.$
11.  $\frac{d^2 y}{dx^2} = 9y; \quad y = \cosh(3x).$
12.  $\frac{d^2 y}{dx^2} = 9y; \quad y = \sinh(3x).$
13.  $\frac{d^2 y}{dx^2} = y^3(1 + 3x^2 y^2); \quad y = (2.75^2 - x^2)^{-1/2}.$
14.  $\frac{d^2 y}{dx^2} = y^3(1 + 3x^2 y^2); \quad y = (2.85^2 - x^2)^{-1/2}.$
15.  $\frac{d^2 y}{dx^2} = 2y^2(1 + 4x^2 y); \quad y = (2.77^2 - x^2)^{-1}.$

$$16. \quad \frac{d^2y}{dx^2} = 2y^2(1 + 4x^2y); \quad y = (2.85^2 - x^2)^{-1}.$$

$$17. \quad \frac{d^2y}{dx^2} = 6x^4y^2(5 + 12x^6y); \quad y = (2.85^6 - x^6)^{-1}.$$

$$18. \quad \frac{d^2y}{dx^2} = 3x^4y^3(5 + 9x^6y^2); \quad y = (2.9^6 - x^6)^{-1/2}.$$

$$19. \quad \frac{d^2y}{dx^2} = -\frac{4y(1 + 6x^2y)}{x^2+0.9} + \frac{24x^2y}{(x^2+0.9)^2} + 2y^2(1 + 4x^2y);$$

$$y = (2.02^2 - x^2)^{-1} + (x^2 + 0.9)^{-1}.$$

$$20. \quad \frac{d^2y}{dx^2} = 2y^2(2x^2 - 3.1804) + 8x^2y^3(2x^2 - 3.1804)^2 + 8x^2y^2;$$

$$y = (2.02^2 - x^2)^{-1}(x^2 + 0.9)^{-1}.$$

$$21. \quad \frac{d^2y}{dx^2} = y^3(2x^2 - 3.1804) + 3x^2y^5(2x^2 - 3.1804)^2 + 4x^2y^3;$$

$$y = (2.02^2 - x^2)^{-1/2}(x^2 + 0.9)^{-1/2}.$$

$$22. \quad \frac{d^2y}{dx^2} = y^3(1+3x^2y^2) - \frac{3y^2(1+5x^2y^2)}{(x^2+0.9)^{1/2}} + \frac{3y(1+10x^2y^2)}{x^2+0.9} - \frac{2(1+15x^2y^2)}{(x^2+0.9)^{3/2}}$$

$$+ \frac{15x^2y}{(x^2+0.9)^2}; \quad y = (2.02^2 - x^2)^{-1/2} + (x^2 + 0.9)^{-1/2}.$$

$$23. \quad \frac{d^2y}{dx^2} = -\frac{2(e^{-y} + 1)}{x^2 + 0.9} + \frac{4x^2(1 - e^{-2y})}{(x^2 + 0.9)^2}; \quad y = \log \left( \frac{2.02 - x^2}{x^2 + 0.9} \right)^2.$$

$$24. \quad \frac{d^2y}{dx^2} = 27x^{10}y^5(x^2+0.9)^4 + 15x^4y^3(x^2+0.9)^2 - 12x^6y^3(x^2+0.9)$$

$$- \frac{2y}{x^2+0.9} + \frac{8x^2y}{(x^2+0.9)^2}; \quad y = \frac{(2.02^6 - x^6)^{-1/2}}{x^2+0.9}.$$

The equations 1-9 have solutions with singularities of various kinds near the boundary of the domain of  $H^2(C_r)$  space with  $r=2.01$ . The solutions of equations 10-12 are entire functions of fast growth. The equations 13-18 have solutions with singularities near

the boundary of the domain of  $H^2(C_r)$  space with  $r=2.8$ . The equations 19-24 have solutions with singularities inside the domain of the space  $H^2(C_r)$  with  $r=2.01$ .

### 4.3 Optimal Multistep Methods in $H^2(C_r)$ -Space

In  $H^2(C_r)$  space, to determine the optimal coefficients  $\hat{\beta}_{1,n}$ ,  $i = \delta_{t_0}(1)k$  of  $\beta$ -optimal multistep method

$$(7) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} f_{n+k-j} = 0$$

where  $\alpha_j$ 's are prefixed according to some consistent and stable known usual method with highest degree polynomial precision,

$$(8) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0,$$

with reference to equation (11) of Chapter 1, we have the system of normal equations given by

$$\hat{C} \hat{b} = \hat{d},$$

where  $\hat{b} = h^2 (\hat{\beta}_{\delta_{t_0},n}, \dots, \hat{\beta}_{k,n})^T$ ,

$$\hat{C}_{1j} = \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3} \left[ 1 + \frac{6x_{n+k-j} \bar{x}_{n+k-1}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})} + \frac{6x_{n+k-j}^2 \bar{x}_{n+k-1}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2} \right],$$

for  $i, j = \delta_{t_0}(1)k$ ,

$$\hat{d}_1 = -\frac{r}{\pi} \sum_{j=0}^k \alpha_j \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3}, \quad i = \delta_{t_0}(1)k.$$

This system of equations may be solved for  $h^2 \hat{\beta}_{j,n}$ ,  $j = \delta_{t_0}(1)k$ .

The following theorem characterizes the  $\beta$ -optimal multistep method (7) in  $H^2(C_r)$  space.

Theorem 3: The optimal multistep method (7) in which  $\alpha_j$ 's are prefixed, and optimization is done with respect to  $\beta_j$ 's in  $H^2(C_r)$  space is characterized by that it is locally interpolatory for functions

$$\left\{ y_i(x) = \frac{x^2}{(r^2 - x\bar{x}_{n+k-1})^3}, \quad i = \delta_{t_0}(1)k \right\}.$$

Proof: The proof follows from Theorem 1 of Chapter 1, using (4) (5) and (6), the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$  and  $D_2''(t, \bar{z})$ .

Corollary 1: If  $x_{n+k-1} = 0$ , for some  $i = \delta_{t_0}(1)k$  then the optimal method (7) becomes consistent.

Proof: Since the corresponding usual formula (8) is consistent, we have  $\sum_{j=0}^k \alpha_j = 0$ ,  $\sum_{j=0}^k j\alpha_j = 0$ ; which also holds for the  $\beta$ -optimal method (7). In other words, the optimal method (7) is exact for constants and  $y(x) = x$ . Furthermore, if  $x_{n+k-1} = 0$ , for some  $i = \delta_{t_0}(1)k$ , then by Theorem 3, the optimal method (7) is exact for  $y(x) = x^2$ . Hence the optimal method (7) is consistent.

In the following tables we are presenting the numerical results for various optimal methods.

Table - 1a

$x_n$	$h^2 \hat{\beta}_{1n}$	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.1064E-01	.4190E-04	.2339E-04
-1.6	.1035E-01	.3253E-05	.1846E-05
-1.5	.1022E-01	.4566E-06	.2677E-06
-1.4	.1015E-01	.9279E-07	.5665E-07
-1.3	.1010E-01	.2433E-07	.1553E-07
-1.2	.1008E-01	.7695E-08	.5154E-08
-1.1	.1006E-01	.2811E-08	.1979E-08
-1.0	.1004E-01	.1153E-08	.8546E-09
-.9	.1003E-01	.5201E-09	.4062E-09
-.8	.1002E-01	.2546E-09	.2093E-09
-.7	.1002E-01	.1339E-09	.1157E-09
-.6	.1001E-01	.7532E-10	.6807E-10
-.5	.1001E-01	.4518E-10	.4245E-10
-.4	.1001E-01	.2899E-10	.2807E-10
-.3	.1000E-01	.2004E-10	.1979E-10
-.2	.1000E-01	.1512E-10	.1507E-10
-.1	.1000E-01	.1264E-10	.1264E-10
.0	.1000E-01	.1189E-10	.1189E-10
.1	.1000E-01	.1264E-10	.1264E-10
.2	.1000E-01	.1512E-10	.1507E-10
.3	.1000E-01	.2004E-10	.1979E-10
.4	.1001E-01	.2899E-10	.2807E-10
.5	.1001E-01	.4518E-10	.4245E-10
.6	.1001E-01	.7532E-10	.6807E-10
.7	.1002E-01	.1339E-09	.1157E-09
.8	.1002E-01	.2546E-09	.2093E-09
.9	.1003E-01	.5201E-09	.4062E-09
1.0	.1004E-01	.1153E-08	.8546E-09
1.1	.1006E-01	.2811E-08	.1979E-08
1.2	.1008E-01	.7695E-08	.5154E-08
1.3	.1010E-01	.2433E-07	.1553E-07
1.4	.1015E-01	.9279E-07	.5665E-07
1.5	.1022E-01	.4566E-06	.2677E-06
1.6	.1035E-01	.3253E-05	.1846E-05
1.7	.1064E-01	.4190E-04	.2339E-04

Optimal  $\beta$  and the square norm of local truncation error functionals for 1-point Störmer's usual and  $\beta$ -optimal methods in  $H^2(C_r)$  -space, for 24 BVP-s, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_1=1$ .

The above table shows that numerically  $\hat{\beta}_{1n}$ 's are dependent on points. At a point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ , and in a neighbourhood of the origin,  $\hat{\beta}_{1n}$ 's are close to usual  $\beta_1$ ,  $\|\hat{T}_n\|^2$  and  $\|T_n\|^2$  are very close.

Table - 1b

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	3	.275334E-02	3	.745402E-03
2	3	.137251E-02	2	.131767E-03
3	3	.425201E-02	3	.171838E-02
4	3	.226162E-02	2	.393388E-03
5	3	.547598E-02	2	.391630E-03
6	3	.110208E-02	3	.512200E-03
7	3	.269263E-03	3	.127965E-03
8	3	.152372E-02	3	.451638E-03
9	3	.829285E-03	3	.275791E-03
10	1	.122695E+00	1	.897968E-01
11	1	.163739E+00	1	.715156E-01
12	1	.159277E+00	1	.678948E-01
13	2	.262515E-03	2	.116181E-03
14	2	.190524E-03	2	.655540E-04
15	2	.279694E-03	2	.158166E-03
16	2	.207536E-03	2	.104474E-03
17	2	.237192E-05	2	.143541E-05
18	2	.200635E-04	2	.108095E-04
19	3	.486991E-02	3	.191473E-02
20	4	.601963E-02	3	.295464E-02
21	3	.473819E-02	3	.151276E-02
22	3	.268350E-02	3	.558582E-03
23	3	.474838E-02	2	.856693E-03
24	3	.924705E-03	2	.320007E-03

Table for number of iterations for convergence of the solution and the average discretisation error using Stormer's 1-point usual and  $\beta$ -optimal methods in  $H^2(C_r)$ -space, for 24 BVP-s.

The results in Table-1b are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$  for the equations 1-9, 18-24; and with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$  for the equations 10-18. The optimal method is one decimal place better on equations 1,2,4-6,8,10-12, 14,22,23; and just better on equations 3,7,9,13,15-18,19,20,21,24.

Table - 2a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.6405E-03	.9069E-02	.1461E-03	.9855E-02	.1102E-05	.9991E-07
-1.6	.7134E-03	.8720E-02	.5246E-03	.9958E-02	.2478E-07	.2731E-08
-1.5	.7530E-03	.8567E-02	.6637E-03	.9984E-02	.1375E-08	.1808E-09
-1.4	.7770E-03	.8487E-02	.7292E-03	.9993E-02	.1326E-09	.2075E-10
-1.3	.7926E-03	.8439E-02	.7650E-03	.9996E-02	.1860E-10	.3476E-11
-1.2	.8033E-03	.8408E-02	.7866E-03	.9998E-02	.3431E-11	.7689E-12
-1.1	.8109E-03	.8387E-02	.8006E-03	.9999E-02	.7802E-12	.2106E-12
-1.0	.8166E-03	.8373E-02	.8101E-03	.9999E-02	.2094E-12	.6830E-13
-.9	.8209E-03	.8362E-02	.8168E-03	.1000E-01	.6440E-13	.2536E-13
-.8	.8242E-03	.8354E-02	.8217E-03	.1000E-01	.2219E-13	.1050E-13
-.7	.8267E-03	.8348E-02	.8253E-03	.1000E-01	.8485E-14	.4806E-14
-.6	.8287E-03	.8343E-02	.8279E-03	.1000E-01	.3576E-14	.2396E-14
-.5	.8303E-03	.8340E-02	.8298E-03	.1000E-01	.1626E-14	.1254E-14
-.4	.8314E-03	.8337E-02	.8312E-03	.1000E-01	.8270E-15	.7184E-15
-.3	.8323E-03	.8336E-02	.8322E-03	.1000E-01	.4759E-15	.4490E-15
-.2	.8329E-03	.8334E-02	.8328E-03	.1000E-01	.2995E-15	.2949E-15
-.1	.8332E-03	.8334E-02	.8332E-03	.1000E-01	.2026E-15	.2023E-15
.0	.8333E-03	.8333E-02	.8333E-03	.1000E-01	.1766E-15	.1766E-15
.1	.8332E-03	.8334E-02	.8332E-03	.1000E-01	.1319E-15	.1316E-15
.2	.8328E-03	.8334E-02	.8329E-03	.1000E-01	.2288E-15	.2242E-15
.3	.8322E-03	.8336E-02	.8323E-03	.1000E-01	.4405E-15	.4137E-15
.4	.8312E-03	.8337E-02	.8314E-03	.1000E-01	.8094E-15	.7008E-15
.5	.8298E-03	.8340E-02	.8303E-03	.1000E-01	.1608E-14	.1236E-14
.6	.8279E-03	.8343E-02	.8287E-03	.1000E-01	.3523E-14	.2342E-14
.7	.8253E-03	.8348E-02	.8267E-03	.1000E-01	.8555E-14	.4877E-14
.8	.8217E-03	.8354E-02	.8242E-03	.1000E-01	.2230E-13	.1061E-13
.9	.8168E-03	.8362E-02	.8209E-03	.1000E-01	.6440E-13	.2536E-13
1.0	.8101E-03	.8373E-02	.8166E-03	.9999E-02	.2094E-12	.6825E-13
1.1	.8006E-03	.8387E-02	.8109E-03	.9999E-02	.7802E-12	.2106E-12
1.2	.7866E-03	.8408E-02	.8033E-03	.9998E-02	.3431E-11	.7690E-12
1.3	.7650E-03	.8439E-02	.7926E-03	.9996E-02	.1860E-10	.3476E-11
1.4	.7292E-03	.8487E-02	.7770E-03	.9993E-02	.1326E-09	.2075E-10
1.5	.6637E-03	.8567E-02	.7530E-03	.9984E-02	.1375E-08	.1808E-09
1.6	.5246E-03	.8720E-02	.7134E-03	.9958E-02	.2478E-07	.2731E-08
1.7	.1461E-03	.9069E-02	.6405E-03	.9855E-02	.1102E-05	.9991E-07

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals using 3-points Cowell's usual and the corresponding  $\beta$ -optimal methods in  $H^2(C_r)$ -space, at the nodal points, with  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ,  $r=2.01$   $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ .

From Table 2a, it is seen that  $\hat{\beta}_i$ 's are dependent on points, in a neighbourhood of the origin they are very close to usual  $\beta_i$ 's. At a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$  and in a neighbourhood of the

origin  $x_n=0$ ,  $\|\hat{T}_n\|^2$  and  $\|T_n\|^2$  are nearly equal.

In the list of 24 differential equations, various equations need various ranges of intervals for the optimal methods to show better performance than the usual method. For convenience of presentation in tabular form we compromise in this regard. We take a common domain of the space  $H^2(C_r)$  and the interval of IVP or BVP for a class of differential equations for each of which the optimal methods perform just better or better than the usual method. The exceptional cases are discussed separately.

In Table 2b(i) we present the numerical results obtained from 24 BVP's in  $H^2(C_r)$ -space with  $r=2.01$   $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ . The optimal method produces worse results than the usual method while implemented on the equations 10-18. This phenomena happens because the solutions of the equations 10-12 are entire functions and the solutions of the equations 13-18 have singularities far from the boundary of the domain of  $H^2(C_r)$ -space with  $r=2.01$ . However, with increasing domain of the space  $H^2(C_r)$  and with increasing interval  $[a,b]$  viz. with  $r=2.8$   $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$ ,  $\beta$ - optimal method performs better than the usual method and these results are shown in Table 2b(ii). Similar cause and effect are also observed in other optimal methods discussed latter. Tables 2b(i) and 2b(ii) reveal that as compared to the usual method, the  $\beta$ -optimal method for BVP is two decimal places better on equations 4, 21, 22; one decimal place better on equations 1,3,5-14,16-20,23; and just better on equations 2,15,24.



Table - 2b(i)

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.156144E-04
2	2	.610703E-04	2	.178799E-04
3	3	.430195E-03	2	.479332E-04
4	2	.133292E-03	2	.657091E-05
5	2	.348263E-03	2	.659629E-04
6	3	.127772E-03	2	.198516E-04
7	2	.313175E-04	2	.511442E-05
8	2	.133523E-03	2	.104334E-04
9	2	.735570E-04	2	.794383E-05
10	1	.506252E-03	1	.392665E-03
11	1	.351768E-03	1	.459791E-02
12	1	.331492E-03	1	.453892E-02
13	2	.698086E-06	2	.165936E-04
14	2	.451879E-06	2	.152147E-04
15	2	.869978E-06	2	.113683E-04
16	2	.592198E-06	2	.104150E-04
17	1	.544969E-08	1	.321212E-07
18	1	.382493E-07	1	.344680E-06
19	3	.509406E-03	2	.496179E-04
20	3	.464907E-03	2	.452461E-04
21	2	.304135E-03	2	.292641E-05
22	2	.208304E-03	2	.616719E-05
23	2	.300455E-03	2	.758440E-04
24	2	.287372E-04	2	.173342E-04

Table - 2b(ii)

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
10	1	.170782E-02	1	.246413E-03
11	1	.738872E-03	1	.659951E-04
12	1	.718937E-03	1	.566636E-04
13	2	.298705E-05	2	.553934E-06
14	2	.167071E-05	1	.121468E-06
15	2	.392662E-05	2	.103090E-05
16	2	.236517E-05	2	.469570E-06
17	1	.257920E-07	1	.793687E-08
18	1	.151826E-06	1	.478944E-07

Tables for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $H^2(C_r)$ -space, for 24 BVP-s.

Table - 2c

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.967316E-03
2	4	.250457E-02	3	.554172E-03
3	8	.760039E-01	5	.797436E-02
4	5	.132805E-01	3	.154057E-03
5	4	.716823E-02	4	.139008E-02
6	5	.356402E-02	4	.585207E-03
7	4	.354625E-03	3	.650670E-04
8	4	.211573E-02	3	.185227E-03
9	3	.604542E-03	3	.843654E-04
10	4	.119641E-01	4	.577624E-02
11	4	.130464E+00	4	.159482E-01
12	4	.129361E+00	4	.158529E-01
13	2	.128426E-05	1	.249427E-06
14	2	.831124E-06	1	.817898E-07
15	2	.193794E-05	2	.612107E-06
16	2	.130401E-05	2	.344920E-06
17	1	.456442E-08	1	.932196E-09
18	1	.343343E-07	1	.276732E-08
19	11	.103867E+00	5	.520553E-02
20	4	.188589E-02	3	.128487E-03
21	4	.112364E-02	3	.630115E-04
22	5	.115497E-01	4	.746171E-03
23	4	.610397E-02	3	.112045E-02
24	3	.542588E-04	3	.660853E-04

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $H^2(C_r)$ -space, for 24 IVP-s.

In Table 2c, we present the numerical results on 24 IVP's. The performance of the optimal method on the IVP's 10-18 is the same as in Table 2b. So, for IVP's 1-9 and 19-24 we take  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$ ; and for the IVP's 10-18 we take  $r=2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$ . Still for the IVP 24, the optimal method is giving little worse result compared to the usual method in the interval  $a=-1.7$ ,  $b=1.7$ . But with an increasing interval, say  $a=-1.8$ ,  $b=1.8$  and with  $r=2.01$ ,  $n=36$ ,  $h=.1$ , the optimal method for IVP 24 gives just better result compared to the usual method, given by  $itu=24$ ,  $\bar{e}_u=.404924E-03$ ,  $ito=3$ ,  $\bar{e}_o=.200161E-03$ .

From Table 2c, we see that as compared to the usual method, the  $\beta$ -optimal method for IVP is two decimal places better on equations 1,4,19,21,22; one decimal place better on equations 2,3, 6-18,20; and just better on equations 5,23,24.

Table - 3a

$x_n$	$h^2\hat{\beta}_{0n}$	$h^2\hat{\beta}_{1n}$	$h^2\hat{\beta}_{2n}$	$h^2\hat{\beta}_{3n}$	$h^2\hat{\beta}_{4n}$
-1.3	.1014E-01	-.6255E-03	.5450E-03	-.9355E-04	.5185E-05
-1.2	.1045E-01	-.1234E-02	.9636E-03	-.2122E-03	.1663E-04
-1.1	.1071E-01	-.1811E-02	.1426E-02	-.3695E-03	.3560E-04
-1.0	.1093E-01	-.2350E-02	.1908E-02	-.5550E-03	.6152E-04
-.9	.1112E-01	-.2851E-02	.2394E-02	-.7610E-03	.9346E-04
-.8	.1129E-01	-.3319E-02	.2879E-02	-.9815E-03	.1305E-03
-.7	.1144E-01	-.3756E-02	.3359E-02	-.1213E-02	.1719E-03
-.6	.1157E-01	-.4167E-02	.3830E-02	-.1452E-02	.2169E-03
-.5	.1169E-01	-.4556E-02	.4294E-02	-.1697E-02	.2652E-03
-.4	.1180E-01	-.4925E-02	.4751E-02	-.1947E-02	.3166E-03
-.3	.1191E-01	-.5280E-02	.5205E-02	-.2204E-02	.3709E-03
-.2	.1201E-01	-.5624E-02	.5657E-02	-.2468E-02	.4286E-03
-.1	.1210E-01	-.5962E-02	.6114E-02	-.2742E-02	.4901E-03
.0	.1219E-01	-.6298E-02	.6579E-02	-.3028E-02	.5560E-03
.1	.1228E-01	-.6637E-02	.7058E-02	-.3330E-02	.6273E-03
.2	.1237E-01	-.6982E-02	.7559E-02	-.3653E-02	.7053E-03
.3	.1246E-01	-.7340E-02	.8090E-02	-.4002E-02	.7917E-03
.4	.1255E-01	-.7716E-02	.8660E-02	-.4387E-02	.8889E-03
.5	.1265E-01	-.8117E-02	.9283E-02	-.4816E-02	.9999E-03
.6	.1275E-01	-.8553E-02	.9975E-02	-.5304E-02	.1129E-02
.7	.1286E-01	-.9035E-02	.1076E-01	-.5870E-02	.1282E-02
.8	.1299E-01	-.9577E-02	.1166E-01	-.6540E-02	.1468E-02
.9	.1312E-01	-.1020E-01	.1273E-01	-.7354E-02	.1700E-02
1.0	.1328E-01	-.1094E-01	.1403E-01	-.8369E-02	.1997E-02
1.1	.1346E-01	-.1182E-01	.1565E-01	-.9676E-02	.2392E-02
1.2	.1368E-01	-.1292E-01	.1773E-01	-.1142E-01	.2938E-02
1.3	.1394E-01	-.1434E-01	.2053E-01	-.1386E-01	.3732E-02
1.4	.1428E-01	-.1624E-01	.2447E-01	-.1746E-01	.4956E-02
1.5	.1473E-01	-.1893E-01	.3040E-01	-.2319E-01	.7004E-02
1.6	.1537E-01	-.2306E-01	.4017E-01	-.3327E-01	.1084E-01
1.7	.1636E-01	-.3013E-01	.5859E-01	-.5396E-01	.1933E-01
1.8	.1810E-01	-.4470E-01	.1018E+00	-.1083E+00	.4393E-01

Table for optimal  $\beta$ 's in  $H^2(C_r)$ -space using  $\beta$ -optimal method, corresponding to Stormer's 5-points explicit method, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=1.2458$ ,  $\beta_1=-.73333$ ,  $\beta_2=.80833$ ,  $\beta_3=-.4$ ,  $\beta_4=.079167$ .

Table - 3b

$x$	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.200	.422951E-06	.204586E-10
-1.100	.285004E-07	.441966E-11
-1.000	.303227E-08	.122238E-11
-.900	.444703E-09	.414708E-12
-.800	.829548E-10	.166763E-12
-.700	.186873E-10	.773688E-13
-.600	.490718E-11	.406314E-13
-.500	.146524E-11	.237813E-13
-.400	.488836E-12	.153025E-13
-.300	.180134E-12	.107809E-13
-.200	.729000E-13	.825421E-14
-.100	.324572E-13	.681878E-14
.000	.161350E-13	.605826E-14
.100	.934011E-14	.581015E-14
.200	.671864E-14	.593563E-14
.300	.663617E-14	.663170E-14
.400	.895454E-14	.800753E-14
.500	.151874E-13	.103936E-13
.600	.301961E-13	.146960E-13
.700	.670408E-13	.225418E-13
.800	.164070E-12	.381147E-13
.900	.440959E-12	.717456E-13
1.000	.130761E-11	.152142E-12
1.100	.432537E-11	.369664E-12
1.200	.162302E-10	.105397E-11
1.300	.707462E-10	.364126E-11
1.400	.370494E-09	.159507E-10
1.500	.244802E-08	.946202E-10
1.600	.219908E-07	.840254E-09
1.700	.303696E-06	.131715E-07

Table for square norm of local truncation error functionals in  $H^2(C_r)$ -space, for Stormer's 5-points explicit method and the corresponding  $\beta$ -optimal method at the nodal points, with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$ .

From Table 3a, it is seen that the optimal coefficients  $\hat{\beta}_{1n}$ ,  $i = 0(1)4$  are varying from point to point and at a neighbourhood of  $x=0$ , they are close to the usual coefficients  $\beta_i$ ,  $i = 0(1)4$ .

Table 3b, displays that at a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$  and at the beginning and the end of the table, the ratios of  $\|\hat{T}_n\|^2$  to  $\|T_n\|^2$  are much less than that at the middle of the table.

Table - 3c

eqn.	$\bar{e}_u$	$\bar{e}_0$
1	.304356E-01	.390952E-04
2	.632144E-02	.655918E-04
3	.167084E+00	.987157E-04
4	.289167E-01	.611449E-04
5	.186264E-01	.252343E-03
6	.748945E-02	.210860E-04
7	.854658E-03	.302262E-05
8	.515672E-02	.162121E-04
9	.163408E-02	.659739E-05
10	.402219E+02	.838737E-02
11	.670431E+02	.633130E+00
12	.670732E+02	.503651E+00
13	.635226E-03	.170320E-05
14	.266670E-03	.242168E-05
15	.152962E-02	.141532E-04
16	.700262E-03	.243861E-05
17	.148635E-05	.104435E-07
18	.672210E-05	.343957E-08
19	.246038E+00	.342962E-03
20	.415804E-02	.202398E-04
21	.282785E-02	.369438E-04
22	.253443E-01	.332677E-03
23	.161435E-01	.118800E-02
24	.151126E-03	.115419E-04

Table for number of iterations for convergence of the solution and the average discretisation error using Stormer's 5-points usual and the corresponding  $\beta$ -optimal methods in  $H^2(C_r)$ -space, for 24 IVP-s.

In Table 3c, the results for IVP's with equations 1-9 and 19-24 are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$ ; and for IVP's with equations 10-18 are obtained with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$ . As compared to usual method, the  $\beta$ -optimal method for IVP is four decimal places better on equations 3,10; three decimal places better on equations 1,4,9,18,19; two decimal places better on equations 2,5-8,11-17,20-22; and one decimal place better on equations 23,24.

In  $H^2(C_r)$  space, to determine the optimal coefficients  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$  of an  $\alpha$ -optimal multistep method

$$(9) \quad \sum_{j=0}^k \hat{\alpha}_{j,n} y_{n+k-j} + h^2 \sum_{j=\delta_{t0}}^k \beta_j f_{n+k-j} = 0,$$

where  $\beta_j$ 's are prefixed according to some consistent and stable known usual method with highest degree polynomial precision. With reference to equation (13) of Chapter 1, we have the system of normal equations given by

$$\hat{C} \hat{a} = \hat{d},$$

where  $\hat{a} = \left( \hat{\alpha}_{0,n}, \dots, \hat{\alpha}_{k,n} \right)^T$ ,

$$\hat{C}_{ij} = \frac{r}{2\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_{n+k-i})}, \quad i, j = 0(1)k, \text{ and}$$

$$\hat{d}_i = - \frac{r}{\pi} h^2 \sum_{j=\delta_{t0}}^k \beta_j \frac{\bar{x}_{n+k-i}^2}{(r^2 - \bar{x}_{n+k-i} x_{n+k-j})^3}, \quad i = 0(1)k.$$

This system of equations can be solved for  $\hat{\alpha}_{j,n}$ ,  $j = 0(1)k$ .

The following theorem characterizes the  $\alpha$ -optimal multistep method (9) in  $H^2(C_r)$  space.

Theorem 4: The  $\alpha$ -optimal multistep method (9) in which  $\beta_j$ 's are prefixed and optimization is done with respect to  $\alpha_j$ 's in  $H^2(C_r)$  space, is characterized by that it is locally interpolatory for functions

$$\left\{ y_i(x) = \frac{1}{(r^2 - x \bar{x}_{n+k-i})}, \quad i = 0(1)k \right\}.$$

Proof: The proof follows from Theorem 2 of Chapter 1, using the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$  and  $D_2''(t, \bar{z})$  from (4), (5) and (6).

Remark: If  $x_{n+k-1} = 0$ , for some  $i = 0(1)k$ , then the  $\alpha$ -optimal method (9) is locally interpolatory for constants, which in other words tells that  $\sum_{j=0}^k \hat{\alpha}_{j,n} = 0$ .

Table - 4a

$x_n$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\alpha}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.5	-.1000E+01	.2001E+01	-.1001E+01	-.3515E-05	.1629E-09	.1441E-09
-1.4	-.1000E+01	.2001E+01	-.1000E+01	-.1078E-05	.2208E-10	.2007E-10
-1.3	-.1000E+01	.2000E+01	-.1000E+01	-.3659E-06	.3966E-11	.3687E-11
-1.2	-.1000E+01	.2000E+01	-.1000E+01	-.1328E-06	.8822E-12	.8363E-12
-1.1	-.1000E+01	.2000E+01	-.1000E+01	-.5014E-07	.2323E-12	.2239E-12
-1.0	-.1000E+01	.2000E+01	-.1000E+01	-.1926E-07	.7020E-13	.6847E-13
-.9	-.1000E+01	.2000E+01	-.1000E+01	-.7369E-08	.2378E-13	.2337E-13
-.8	-.1000E+01	.2000E+01	-.1000E+01	-.2747E-08	.8889E-14	.8876E-14
-.7	-.1000E+01	.2000E+01	-.1000E+01	-.9720E-09	.3630E-14	.3688E-14
-.6	-.1000E+01	.2000E+01	-.1000E+01	-.3154E-09	.1623E-14	.1655E-14
-.5	-.1000E+01	.2000E+01	-.1000E+01	-.8929E-10	.7591E-15	.7524E-15
-.4	-.1000E+01	.2000E+01	-.1000E+01	-.2032E-10	.3438E-15	.3307E-15
-.3	-.1000E+01	.2000E+01	-.1000E+01	-.3179E-11	.2282E-15	.2356E-15
-.2	-.1000E+01	.2000E+01	-.1000E+01	-.2334E-12	.1914E-15	.2013E-15
-.1	-.1000E+01	.2000E+01	-.1000E+01	.0000E+00	.1079E-15	.9421E-16
.0	-.1000E+01	.2000E+01	-.1000E+01	.2220E-15	.1088E-15	.1178E-15
.1	-.1000E+01	.2000E+01	-.1000E+01	-.2220E-15	.3723E-16	.4516E-16
.2	-.1000E+01	.2000E+01	-.1000E+01	-.2371E-12	.5005E-16	.3525E-16
.3	-.1000E+01	.2000E+01	-.1000E+01	-.3182E-11	.1575E-15	.1388E-15
.4	-.1000E+01	.2000E+01	-.1000E+01	-.2031E-10	.3880E-15	.3597E-15
.5	-.1000E+01	.2000E+01	-.1000E+01	-.8928E-10	.7591E-15	.7507E-15
.6	-.1000E+01	.2000E+01	-.1000E+01	-.3154E-09	.1659E-14	.1625E-14
.7	-.1000E+01	.2000E+01	-.1000E+01	-.9720E-09	.3577E-14	.3539E-14
.8	-.1000E+01	.2000E+01	-.1000E+01	-.2747E-08	.8872E-14	.8818E-14
.9	-.1000E+01	.2000E+01	-.1000E+01	-.7369E-08	.2380E-13	.2347E-13
1.0	-.1000E+01	.2000E+01	-.1000E+01	-.1926E-07	.7025E-13	.6849E-13
1.1	-.1000E+01	.2000E+01	-.1000E+01	-.5014E-07	.2325E-12	.2239E-12
1.2	-.1000E+01	.2000E+01	-.1000E+01	-.1328E-06	.8822E-12	.8363E-12
1.3	-.1000E+01	.2000E+01	-.1000E+01	-.3659E-06	.3966E-11	.3687E-11
1.4	-.1000E+01	.2001E+01	-.1000E+01	-.1078E-05	.2208E-10	.2007E-10
1.5	-.1001E+01	.2001E+01	-.1000E+01	-.3515E-05	.1629E-09	.1441E-09

Table for optimal  $\alpha$ 's and their row-sum and the square norm of local truncation error functionals, in  $H^2(C_r)$ -space, using Cowell's 3-points usual method and the corresponding  $\alpha$ -optimal method, at the nodal points, with  $r=2.1$ ,  $a=-1.6$ ,  $b=1.6$ ,  $n=32$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ;  $\beta_0=1/12$ ,  $\beta_1=10/12$ ,  $\beta_2=1/12$ .

From Table 4a, it is seen that within the tabular precision, the optimal coefficients  $\hat{\alpha}_{0n}$ ,  $\hat{\alpha}_{1n}$ ,  $\hat{\alpha}_{2n}$  are equal to the usual coefficients  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$  and that at a general point  $x_n$ ,  $\|\hat{T}_n\|^2 < \|T_n\|^2$ , except at  $x = 0$  and  $.1$ , where the errors are too small, so that the reversal of the inequality must be because of round-offs. Even though all the numerical calculations are performed with double precision, such small numbers might be affected by round-offs and may not be very reliable.

Table - 4b

eqn	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.128041E-03
2	2	.610703E-04	2	.192381E-04
3	3	.430195E-03	2	.286316E-03
4	2	.133292E-03	2	.544429E-04
5	2	.348263E-03	2	.555233E-03
6	3	.127772E-03	2	.879302E-04
7	2	.313175E-04	2	.202928E-04
8	2	.133523E-03	2	.538413E-04
9	2	.735570E-04	2	.214216E-04
10	1	.170782E-02	1	.141672E-02
11	1	.738872E-03	1	.276783E-03
12	1	.718937E-03	1	.260809E-03
13	2	.298705E-05	2	.220359E-05
14	2	.167071E-05	2	.102236E-05
15	2	.392662E-05	2	.327281E-05
16	2	.236517E-05	2	.185000E-05
17	1	.257920E-07	1	.170986E-07
18	1	.151826E-06	1	.331515E-07
19	3	.509406E-03	2	.285533E-03
20	3	.464907E-03	2	.257529E-03
21	2	.304135E-03	2	.264234E-04
22	2	.208304E-03	2	.192195E-04
23	2	.300455E-03	2	.637374E-03
24	2	.287372E-04	2	.487327E-04

Table for number of iterations for convergence of the solution and the average discretisation error using Cowell's 3-points usual and the corresponding  $\alpha$ -optimal methods in  $H^2(C_r)$ -space, for 24 BVP-s.



The results in Table 4b, for BVP's with equations 1-9 and 19-24 are obtained with  $r=2.1$ ,  $a=-1.6$ ,  $b=1.6$ ,  $n=32$ ,  $h=.1$ ; and the results for BVP's with equations 10-18 are obtained with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$ . It is seen from the table that the  $\alpha$ -optimal method is producing worse results compared to the usual method for the equations 23 and 24 and this situation is not being improved with the same  $r$  and  $h$ , in the intervals  $[-1.8,1.8]$ ,  $[-1.7,1.7]$ ,  $[-1.5,1.5]$ , ...,  $[-.8,.8]$ . The equation 23 is giving just better result in the interval  $[-.7, .7]$ , but the equation 24 still does not do so. Ultimately, in the interval  $[-.5, .5]$  both of them are giving just better results, and these results are given below.

eqn	itu	$\bar{e}_u$	ito	$\bar{e}_o$
23:	1	.445555E-05	1	.444995E-05
24:	1	.158786E-05	1	.158745E-05

The above numerical results show that as compared to the usual method the  $\alpha$ -optimal method is one decimal place better on equations 4,6,8,18,21,22; and just better on equations 1,2,3,5,7, 9-17,19,20,23,24.

In  $H^2(C_r)$  space, to determine the optimal coefficients  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k$  of  $\beta$ -optimal method (7) subject to the condition

$$(10) \quad \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} = 1,$$

where  $\alpha_j$  's are prefixed according to some consistent and stable known usual method (8) with highest degree polynomial precision. With reference to equation (15) of Chapter 1, we have the system of normal equations given by

$$\hat{c} \hat{b} = \hat{d},$$

$$\text{where } \hat{b} = h^2 \left( \hat{\beta}_{\delta_{t_0, n}}, \dots, \hat{\beta}_{k-1, n} \right)^T,$$

$$\begin{aligned} \hat{c}_{ij} = & \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3} \left[ 1 + \frac{6x_{n+k-j} \bar{x}_{n+k-1}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})} + \frac{6x_{n+k-j}^2 \bar{x}_{n+k-1}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2} \right] \\ & - \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-j} \bar{x}_n)^3} \left[ 1 + \frac{6x_{n+k-j} \bar{x}_n}{(r^2 - x_{n+k-j} \bar{x}_n)} + \frac{6x_{n+k-j}^2 \bar{x}_n^2}{(r^2 - x_{n+k-j} \bar{x}_n)^2} \right] \\ & - \frac{r}{\pi} \frac{2}{(r^2 - x_n \bar{x}_{n+k-1})^3} \left[ 1 + \frac{6x_n \bar{x}_{n+k-1}}{(r^2 - x_n \bar{x}_{n+k-1})} + \frac{6x_n^2 \bar{x}_{n+k-1}^2}{(r^2 - x_n \bar{x}_{n+k-1})^2} \right] \\ & + \frac{r}{\pi} \frac{2}{(r^2 - x_n \bar{x}_n)^3} \left[ 1 + \frac{6x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{6x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right], \quad i, j = \delta_{t_0}(1)k-1, \text{ and} \end{aligned}$$

$$\begin{aligned} \hat{d}_i = & - \frac{r}{\pi} \sum_{j=0}^k \alpha_j \left\{ \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3} - \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_n)^3} \right\} \\ & - h^2 \frac{r}{\pi} \left[ \frac{2}{(r^2 - x_n \bar{x}_{n+k-1})^3} \left\{ 1 + \frac{6x_n \bar{x}_{n+k-1}}{(r^2 - x_n \bar{x}_{n+k-1})} + \frac{6x_n^2 \bar{x}_{n+k-1}^2}{(r^2 - x_n \bar{x}_{n+k-1})^2} \right\} \right. \\ & \left. - \frac{2}{(r^2 - x_n \bar{x}_n)^3} \left\{ 1 + \frac{6x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{6x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right\} \right], \quad i = \delta_{t_0}(1)k-1 \end{aligned}$$

This system of equations may be solved for  $h^2 \hat{\beta}_{j, n}$ ,  $j = \delta_{t_0}(1)k-1$ .

The following theorem characterizes the  $\beta$ -optimal multistep method (7) in  $H^2(C_r)$  space subject to the condition (10).

**Theorem 5:** The optimal multistep method (7) in which  $\alpha_j$ 's are prefixed and  $\beta_j$ 's are optimized in  $H^2(C_r)$  space, subject to the condition (10) is characterized by that it is locally interpolatory for functions

$$\left\{ y_i(x) = \frac{x^2}{(r^2 - x \bar{x}_{n+k-1})^3} - \frac{x^2}{(r^2 - x \bar{x}_n)^3}, \quad i = \delta_{t_0}(1)k-1 \right\}.$$

Proof: The proof follows from Theorem 3 of Chapter 1, and the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$  and  $D_2''(t, \bar{z})$  from (4), (5) and (6).

Table - 5a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.6732E-03	.8838E-02	.4892E-03	.1000E-01	.1102E-05	.1568E-06
-1.6	.7375E-03	.8599E-02	.6636E-03	.1000E-01	.2478E-07	.4196E-08
-1.5	.7704E-03	.8496E-02	.7340E-03	.1000E-01	.1375E-08	.2675E-09
-1.4	.7894E-03	.8441E-02	.7692E-03	.1000E-01	.1326E-09	.2932E-10
-1.3	.8014E-03	.8409E-02	.7893E-03	.1000E-01	.1860E-10	.4671E-11
-1.2	.8095E-03	.8389E-02	.8018E-03	.1000E-01	.3431E-11	.9809E-12
-1.1	.8152E-03	.8375E-02	.8102E-03	.1000E-01	.7802E-12	.2550E-12
-1.0	.8195E-03	.8364E-02	.8161E-03	.1000E-01	.2094E-12	.7868E-13
-.9	.8227E-03	.8357E-02	.8204E-03	.1000E-01	.6440E-13	.2794E-13
-.8	.8253E-03	.8351E-02	.8237E-03	.1000E-01	.2219E-13	.1115E-13
-.7	.8273E-03	.8346E-02	.8263E-03	.1000E-01	.8485E-14	.4965E-14
-.6	.8290E-03	.8343E-02	.8284E-03	.1000E-01	.3576E-14	.2431E-14
-.5	.8304E-03	.8340E-02	.8300E-03	.1000E-01	.1626E-14	.1261E-14
-.4	.8314E-03	.8337E-02	.8313E-03	.1000E-01	.8270E-15	.7193E-15
-.3	.8323E-03	.8336E-02	.8322E-03	.1000E-01	.4759E-15	.4491E-15
-.2	.8329E-03	.8334E-02	.8328E-03	.1000E-01	.2995E-15	.2949E-15
-.1	.8332E-03	.8334E-02	.8332E-03	.1000E-01	.2026E-15	.2023E-15
.0	.8333E-03	.8333E-02	.8333E-03	.1000E-01	.1766E-15	.1766E-15
.1	.8332E-03	.8334E-02	.8332E-03	.1000E-01	.1319E-15	.1316E-15
.2	.8328E-03	.8334E-02	.8329E-03	.1000E-01	.2288E-15	.2242E-15
.3	.8322E-03	.8336E-02	.8323E-03	.1000E-01	.4405E-15	.4138E-15
.4	.8313E-03	.8337E-02	.8314E-03	.1000E-01	.8094E-15	.7017E-15
.5	.8300E-03	.8340E-02	.8304E-03	.1000E-01	.1608E-14	.1243E-14
.6	.8284E-03	.8343E-02	.8290E-03	.1000E-01	.3523E-14	.2378E-14
.7	.8263E-03	.8346E-02	.8273E-03	.1000E-01	.8555E-14	.5036E-14
.8	.8237E-03	.8351E-02	.8253E-03	.1000E-01	.2230E-13	.1126E-13
.9	.8204E-03	.8357E-02	.8227E-03	.1000E-01	.6440E-13	.2794E-13
1.0	.8161E-03	.8364E-02	.8195E-03	.1000E-01	.2094E-12	.7862E-13
1.1	.8102E-03	.8375E-02	.8152E-03	.1000E-01	.7802E-12	.2550E-12
1.2	.8018E-03	.8389E-02	.8095E-03	.1000E-01	.3431E-11	.9809E-12
1.3	.7893E-03	.8409E-02	.8014E-03	.1000E-01	.1860E-10	.4672E-11
1.4	.7692E-03	.8441E-02	.7894E-03	.1000E-01	.1326E-09	.2932E-10
1.5	.7340E-03	.8496E-02	.7704E-03	.1000E-01	.1375E-08	.2675E-09
1.6	.6636E-03	.8599E-02	.7375E-03	.1000E-01	.2478E-07	.4196E-08
1.7	.4892E-03	.8838E-02	.6732E-03	.1000E-01	.1102E-05	.1568E-06

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals using 3-points Cowell's usual method and the corresponding  $\beta$ -optimal method with restriction, at the nodal points, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ;  $\alpha_2=-1$ ;  $\beta_0=1/12$ ,  $\beta_1=10/12$ ,  $\beta_2=1/12$ , in  $H^2(C_r)$  space.

Table 5a shows that the optimal coefficients  $\hat{\beta}_{0n}$ ,  $\hat{\beta}_{1n}$ ,  $\hat{\beta}_{2n}$  are changing from point to point and at a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq$

$\|T_n\|^2$ . In a neighbourhood of  $x=0$ ,  $\hat{\beta}_{1n}$ 's are very close to the usual coefficients  $\beta_1$ 's and  $\|\hat{T}_n\|^2$ ,  $\|T_n\|^2$  are numerically equal.

Table - 5b

eqn	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.682802E-04
2	2	.610703E-04	2	.281577E-04
3	3	.430195E-03	2	.104135E-03
4	2	.133292E-03	2	.362855E-04
5	2	.348263E-03	2	.135079E-03
6	3	.127772E-03	2	.338500E-04
7	2	.313175E-04	2	.835447E-05
8	2	.133523E-03	2	.276090E-04
9	2	.735570E-04	2	.207658E-04
10	1	.170782E-02	1	.309772E-03
11	1	.738872E-03	1	.706424E-04
12	1	.718937E-03	1	.609621E-04
13	2	.298705E-05	2	.114488E-05
14	2	.167071E-05	2	.625330E-06
15	2	.392662E-05	2	.149686E-05
16	2	.236517E-05	2	.833412E-06
17	1	.257920E-07	1	.928588E-08
18	1	.151826E-06	1	.553283E-07
19	3	.509406E-03	2	.132122E-03
20	3	.464907E-03	2	.122836E-03
21	2	.304135E-03	2	.917673E-04
22	2	.208304E-03	2	.627561E-04
23	2	.300455E-03	2	.104856E-03
24	2	.287372E-04	2	.121353E-04

Table for number of iterations for convergence of the solution and the average discretisation error using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods with restriction, in  $H^2(C_r)$ -space, for 24 BVP-s.

The numerical results in Table 5b, on 24 BVP-s for equations 1-9 and 19-24 are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ; and for equations 10-18, with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$ . As compared to usual method,  $\beta$ -optimal method with restriction for BVP is one decimal place better on equations 1,4,6-8,10-12,14,16 - 18, 21,22; and just better on equations 2,3,5,9,13,15,19,20,23 and 24.

Table - 5c

eqn	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.410628E-02
2	4	.250457E-02	4	.101724E-02
3	8	.760039E-01	5	.167955E-01
4	5	.132805E-01	4	.338135E-02
5	4	.716823E-02	4	.290794E-02
6	5	.356402E-02	4	.100650E-02
7	4	.354625E-03	3	.103725E-03
8	4	.211573E-02	4	.639029E-03
9	3	.604542E-03	3	.187586E-03
10	4	.119641E-01	4	.621026E-02
11	4	.130464E+00	4	.193414E-01
12	4	.129361E+00	4	.189977E-01
13	2	.128426E-05	2	.561091E-06
14	2	.831124E-06	1	.347910E-06
15	2	.193794E-05	2	.877730E-06
16	2	.130401E-05	2	.563117E-06
17	1	.456442E-08	1	.814943E-09
18	1	.343343E-07	1	.226087E-08
19	11	.103867E+00	6	.208003E-01
20	4	.188589E-02	3	.499274E-03
21	4	.112364E-02	3	.326909E-03
22	5	.115497E-01	4	.332103E-02
23	4	.610397E-02	4	.235365E-02
24	3	.542588E-04	2	.733719E-05

Table for number of iterations for convergence of the solution and the average discretisation error using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods with restriction, in  $H^2(C_r)$ -space, for 24 IVP-s.

The numerical results of Table 5b on 24 IVP-s, are obtained for equations 1-9 and 19-24, with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$ ; and for equations 10-18, with  $r=2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$ . As compared to usual method,  $\beta$ -optimal method with restriction for IVP is one decimal place better on equations 1,4,8,10-13,15-22 and 24; and just better on equations 2,3,5,6,7,9, 14 and 23.

In  $H^2(C_r)$  space, we have to determine the optimal coefficients

$\hat{\alpha}_{1,n}, i = 0(1)k$  and  $\hat{\beta}_{1,n}, i = \delta_{t_0}(1)k-1$  of an  $(\alpha, \beta)$ -optimal multistep method

$$(11) \quad \sum_{j=0}^k \hat{\alpha}_{j,n} y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} f_{n+k-j} = 0,$$

subject to the condition (10). With reference to equation (18) and (19) of Chapter 1, we have the system of normal equations given by

$$\hat{C} \hat{b} = \hat{d}, \quad \text{where}$$

$$\hat{b} = \left( \hat{\alpha}_{0,n}, \dots, \hat{\alpha}_{k,n}, h^2 \hat{\beta}_{\delta_{t_0},n}, \dots, h^2 \hat{\beta}_{k-1,n} \right)^T;$$

$$\hat{C}_{1j} = \frac{r}{2\pi(r^2 - x_{n+k-j} \bar{x}_{n+k-1})}, \quad i, j = 0(1)k;$$

$$\hat{C}_{1j} = \frac{r \bar{x}_{n+k-1}^{-2}}{\pi(r^2 - \bar{x}_{n+k-1} x_{n+k-j})^3} - \frac{r \bar{x}_{n+k-1}^{-2}}{\pi(r^2 - \bar{x}_{n+k-1} x_n)^3}, \quad i = 0(1)k, \\ j = k+1(1)2k+1-\delta_{t_0};$$

$$\hat{C}_{1j} = \frac{r x_{n+k-j}^2}{\pi(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3} - \frac{r x_{n+k-j}^2}{\pi(r^2 - x_{n+k-j} \bar{x}_n)^3}, \quad i = k+1(1)2k+1-\delta_{t_0} \\ j = 0(1)k;$$

$$\hat{C}_{1j} = \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3} \left[ 1 + \frac{6x_{n+k-j} \bar{x}_{n+k-1}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})} + \frac{6x_{n+k-j}^2 \bar{x}_{n+k-1}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2} \right] \\ - \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-j} \bar{x}_n)^3} \left[ 1 + \frac{6x_{n+k-j} \bar{x}_n}{(r^2 - x_{n+k-j} \bar{x}_n)} + \frac{6x_{n+k-j}^2 \bar{x}_n^2}{(r^2 - x_{n+k-j} \bar{x}_n)^2} \right] \\ - \frac{r}{\pi} \frac{2}{(r^2 - x_n \bar{x}_{n+k-1})^3} \left[ 1 + \frac{6x_n \bar{x}_{n+k-1}}{(r^2 - x_n \bar{x}_{n+k-1})} + \frac{6x_n^2 \bar{x}_{n+k-1}^2}{(r^2 - x_n \bar{x}_{n+k-1})^2} \right] \\ + \frac{r}{\pi} \frac{2}{(r^2 - x_n \bar{x}_n)^3} \left[ 1 + \frac{6x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{6x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right], \quad \text{for } i, j = k+1(1)2k+1-\delta_{t_0};$$

$$\hat{d}_1 = h^2 \frac{r \bar{x}_{n+k-1}^2}{\pi (r^2 - \bar{x}_{n+k-1} x_n)^3}, \quad i = 0(1)k;$$

$$\begin{aligned} \hat{d}_1 = & - \frac{r}{\pi} h^2 \left[ \frac{2}{(r^2 - x_n \bar{x}_{n+k-1})^3} \left\{ 1 + \frac{6x_n \bar{x}_{n+k-1}}{(r^2 - x_n \bar{x}_{n+k-1})} + \frac{6x_n^2 \bar{x}_{n+k-1}^2}{(r^2 - x_n \bar{x}_{n+k-1})^2} \right\} \right. \\ & \left. - \frac{2}{(r^2 - x_n \bar{x}_n)^3} \left\{ 1 + \frac{6x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{6x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right\} \right], \quad \text{for } i=k+1(1)2k+1-\delta_{t_0}. \end{aligned}$$

The following theorem characterizes the  $(\alpha, \beta)$ -optimal method (11) in  $H^2(C_r)$  space subject to the condition (10).

Theorem 6: The optimal multistep method (11) in which  $\alpha_j$ 's as well as  $\beta_j$ 's are optimized in  $H^2(C_r)$  space, subject to the condition (10) is characterized by that it is locally interpolatory for functions  $\{y_{1i}(x), i = \delta_{t_0}(1)k-1\} \cup \{y_{2i}(x), i = 0(1)k-1\}$ ,

$$\text{where } y_{1i}(x) = \frac{x^2}{(r^2 - x \bar{x}_{n+k-1})^3} - \frac{x^2}{(r^2 - x \bar{x}_n)^3}, \quad i = \delta_{t_0}(1)k-1$$

$$\text{and } y_{2i}(x) = \frac{1}{(r^2 - x \bar{x}_{n+k-1})}, \quad i = 0(1)k-1.$$

Proof: The proof follows from Theorem 4 of Chapter 1, and the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$  and  $D_2''(t, \bar{z})$  from (4), (5) and (6).

#### 4.4 Optimal Multistep Methods in $H^2(C_r)$ -Space Interpolatory for Polynomials.

In Chapter 1, we have given an approach for determining the optimal multistep methods interpolatory for polynomials of certain degree, in a general Hilbert space with reproducing kernel function. We know

$$(12) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=0}^{k-\delta_{t_0}} \gamma_j \nabla^j f(x_{n+k-\delta_{t_0}}, y_{n+k-\delta_{t_0}}) = 0$$

represents a general  $k$ -step method for a second order differential equation. A general method interpolatory for polynomials of degree  $q < k + \delta_{t_1} + 1$ , can be written as

$$(13) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=0}^{q-2} \gamma_j^u \nabla^j f(x_{n+k-\delta_{t_0}}, y_{n+k-\delta_{t_0}}) = \\ - h^2 \sum_{j=q-1}^{k-\delta_{t_0}} \gamma_j \nabla^j f(x_{n+k-\delta_{t_0}}, y_{n+k-\delta_{t_0}}),$$

with prefixed  $\alpha_j$ 's such that  $\sum_{j=0}^k \alpha_j = 0$  and  $\sum_{j=0}^k j \alpha_j = 0$ , where  $\alpha_j$ 's and  $\gamma_j^u$ 's are according to the corresponding usual method. The remaining coefficients,  $\gamma_j$ ,  $j = q-1(1)k-\delta_{t_0}$ , can be determined by minimizing the norm of the local truncation error functional of the method over the Hilbert space  $H^2(C_r)$ . Let us denote the optimized coefficients as  $\hat{\gamma}_{j,n}$ ,  $j = q-1(1)k-\delta_{t_0}$ .

The local truncation error for the optimal method (13) is

$$\sum_{j=0}^k \alpha_j y(x_{n+k-j}) + h^2 \sum_{j=0}^{q-2} \gamma_j^u \nabla^j y''(x_{n+k-\delta_{t_0}}) \\ + h^2 \sum_{j=q-1}^{k-\delta_{t_0}} \hat{\gamma}_{j,n} \nabla^j y''(x_{n+k-\delta_{t_0}}).$$

Since,  $\nabla^j y''(x_{n+k-\delta_{t_0}}) = \sum_{i=0}^j (-1)^{i,j} C_i y''(x_{n+k-\delta_{t_0}-i})$ , the representer for  $\nabla^j y''(x_{n+k-\delta_{t_0}})$  in  $H^2(C_r)$  space is  $\sum_{i=0}^j (-1)^{i,j} C_i D_2(t, \bar{x}_{n+k-\delta_{t_0}-i})$ .

The representer for the local truncation error functional of the optimal method in  $H^2(C_r)$ -space is then given by



$$\begin{aligned} \hat{T}_n^q = & \sum_{j=0}^k \bar{\alpha}_j K(t, \bar{x}_{n+k-j}) + h^2 \sum_{j=0}^{q-2} \bar{\gamma}_j^u \sum_{i=0}^j (-1)^{i-j} C_i D_2(t, \bar{x}_{n+k-\delta_{t_0}-i}) \\ & + h^2 \sum_{j=q-1}^{k-\delta_{t_0}} \bar{\gamma}_{j,n} \sum_{i=0}^j (-1)^{i-j} C_i D_2(t, \bar{x}_{n+k-\delta_{t_0}-i}). \end{aligned}$$

Since  $\{K(t, \bar{x}_{n+k-i}), i = 0(1)k; D_2(t, \bar{x}_{n+k-i}), i = 0(1)k\}$  is linearly independent, to minimize  $\|\hat{T}_n^q\|$  with respect to  $\hat{\gamma}_{1,n}$ ,  $\delta(\|\hat{T}_n^q\|)$  following a change  $\delta(\hat{\gamma}_{1,n})$  in  $\hat{\gamma}_{1,n}$ ,  $i = q-1(1)k-\delta_{t_0}$ , is to vanish. Proceeding as earlier we get the following system of normal equations:

$$\begin{aligned} & \sum_{i=0}^1 (-1)^{i-1} C_i \left( D_2(t, \bar{x}_{n+k-\delta_{t_0}-i}), \sum_{j=0}^k \bar{\alpha}_j K(t, \bar{x}_{n+k-j}) \right) \\ & + h^2 \sum_{i=0}^1 (-1)^{i-1} C_i \left( D_2(t, \bar{x}_{n+k-\delta_{t_0}-i}), \sum_{j=0}^{q-2} \bar{\gamma}_j^u \sum_{m=0}^j (-1)^{m-j} C_m D_2(t, \bar{x}_{n+k-\delta_{t_0}-m}) \right) \\ & + h^2 \sum_{i=0}^1 (-1)^{i-1} C_i \left( D_2(t, \bar{x}_{n+k-\delta_{t_0}-i}), \right. \\ & \left. \sum_{j=q-1}^{k-\delta_{t_0}} \bar{\gamma}_{j,n} \sum_{m=0}^j (-1)^{m-j} C_m D_2(t, \bar{x}_{n+k-\delta_{t_0}-m}) \right) = 0, \quad i = q-1(1)k-\delta_{t_0}, \end{aligned}$$

$$\begin{aligned} \text{or, } & \sum_{j=0}^k \alpha_j \sum_{i=0}^1 (-1)^{i-1} C_i D_2(x_{n+k-j}, \bar{x}_{n+k-\delta_{t_0}-i}) \\ & + h^2 \sum_{j=0}^{q-2} \gamma_j^u \sum_{m=0}^j \sum_{i=0}^1 (-1)^{i+m-1} C_i^j C_m D_2''(x_{n+k-\delta_{t_0}-m}, \bar{x}_{n+k-\delta_{t_0}-i}) \\ & + h^2 \sum_{j=q-1}^{k-\delta_{t_0}} \gamma_{j,n} \sum_{m=0}^j \sum_{i=0}^1 (-1)^{i+m-1} C_i^j C_m D_2''(x_{n+k-\delta_{t_0}-m}, \bar{x}_{n+k-\delta_{t_0}-i}) = 0, \end{aligned}$$

$$\text{for } i = q-1(1)k-\delta_{t_0},$$

$$\text{or,} \quad \hat{c} \hat{b} = \hat{d},$$

$$\text{where } \hat{b} = \left( \hat{\gamma}_{q-1,n}, \hat{\gamma}_{q,n}, \dots, \hat{\gamma}_{k-\delta_{t_0},n} \right)^T,$$

$$\hat{c}_{1,j} = h^2 \sum_{m=0}^j \sum_{l=0}^1 (-1)^{1+m-l} C_1^j C_m \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l})^3} \cdot$$

$$\left[ 1 + \frac{6x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l}}{(r^2 - x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l})} + \frac{6x_{n+k-\delta_{t_0}-m}^2 \bar{x}_{n+k-\delta_{t_0}-l}^2}{(r^2 - x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l})^2} \right],$$

$$\hat{d}_1 = - \sum_{j=0}^k \alpha_j \sum_{l=0}^1 (-1)^{1-l} C_1 \frac{rx_{n+k-j}^2}{\pi (r^2 - x_{n+k-j} \bar{x}_{n+k-\delta_{t_0}-l})^3}$$

$$- h^2 \sum_{j=0}^{q-2} \gamma_j^u \sum_{m=0}^j \sum_{l=0}^1 (-1)^{1+m-l} C_1^j C_m \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l})^3} \cdot$$

$$\cdot \left[ 1 + \frac{6x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l}}{(r^2 - x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l})} + \frac{6x_{n+k-\delta_{t_0}-m}^2 \bar{x}_{n+k-\delta_{t_0}-l}^2}{(r^2 - x_{n+k-\delta_{t_0}-m} \bar{x}_{n+k-\delta_{t_0}-l})^2} \right],$$

$$\text{for } i=q-1(1)k-\delta_{t_0}.$$

Thus we get the following theorem.

**Theorem 7:** The optimal multistep method (13) interpolatory for polynomials of degree  $q$  is characterized by that it is locally interpolatory for functions

$$\{x^i; i = 1(1)q\} \cup \{h_i(x); i = q-1(1)k-\delta_{t_0}\},$$

$$\text{where } h_i(x) = \sum_{l=0}^1 (-1)^{1-l} C_1 \frac{x^2}{(r^2 - x \bar{x}_{n+k-\delta_{t_0}-l})^3}.$$

We next discuss an  $\alpha$ -optimal multistep method interpolatory for polynomials of degree  $q$ , in  $H^2(C_r)$ -space, in terms of backward differences

$$(14) \quad \sum_{j=2}^k \gamma_j \nabla^j y_{n+k} + h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0,$$

represents a general  $k$ -step method for a 2-nd order differential equation. A general method interpolatory for polynomials of degree  $q$  can be written as

$$(15) \quad \sum_{j=2}^q \gamma_j^u \nabla^j y_{n+k} + \sum_{j=q+1}^k \gamma_j \nabla^j y_{n+k} + h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0,$$

where the coefficients  $\gamma_j^u$ 's are defined as in the corresponding usual method. The remaining coefficients  $\gamma_j$ ,  $j = q+1(1)k$  can be determined by minimizing the norm of the local truncation error functional of the method over the Hilbert space  $H^2(C_r)$ . Let us denote the optimized coefficients as  $\hat{\gamma}_{j,n}$ ,  $j = q+1(1)k$ .

The representer for the local truncation error functional for the optimal method (15) is given by

$$\begin{aligned} \hat{T}_n^q = & \sum_{j=2}^q \bar{\gamma}_j^u \sum_{i=0}^j (-1)^{i-j} C_i K(t, \bar{x}_{n+k-i}) \\ & + \sum_{j=q+1}^k \bar{\gamma}_{j,n} \sum_{i=0}^j (-1)^{i-j} C_i K(t, \bar{x}_{n+k-i}) + h^2 \sum_{j=\delta_{t_0}}^k \bar{\beta}_j D_2(t, \bar{x}_{n+k-j}). \end{aligned}$$

Since  $S = \{K(t, \bar{x}_{n+k-j}), j=0(1)k; D_2(t, \bar{x}_{n+k-j}), j=0(1)k\}$  is linearly independent, to minimize  $\|\hat{T}_n^q\|$  with respect to  $\hat{\gamma}_{1,n}$ ,  $\delta(\|\hat{T}_n^q\|)$ , following a change  $\delta(\hat{\gamma}_{1,n})$  in  $\hat{\gamma}_{1,n}$ ,  $l = q+1(1)k$ , is to vanish. Proceeding as earlier we get the following normal equations.

$$\begin{aligned}
& \sum_{j=2}^q \gamma_j^u \sum_{m=0}^j \sum_{l=0}^1 (-1)^{l+m-1} C_1^j C_m^l K(x_{n+k-m}, \bar{x}_{n+k-l}) \\
& + \sum_{j=q+1}^k \hat{\gamma}_{j,n} \sum_{m=0}^j \sum_{l=0}^1 (-1)^{l+m-1} C_1^j C_m^l K(x_{n+k-m}, \bar{x}_{n+k-l}) \\
& + h^2 \sum_{j=\delta_{t0}}^k \beta_j \sum_{l=0}^1 (-1)^{l-1} C_1^l \overline{D_2(x_{n+k-l}, \bar{x}_{n+k-j})} = 0, \quad i = q+1(1)k,
\end{aligned}$$

or,  $\hat{C} \hat{a} = \hat{d}$ , where

$$\hat{a} = \left( \hat{\gamma}_{q+1,n}, \dots, \hat{\gamma}_{k,n} \right)^T,$$

$$\hat{C}_{ij} = \sum_{m=0}^j \sum_{l=0}^1 (-1)^{l+m-1} C_1^j C_m^l \frac{r}{2\pi(r^2 - x_{n+k-m} \bar{x}_{n+k-l})}, \quad i, j = q+1(1)k,$$

$$\begin{aligned}
\hat{d}_i = & - \sum_{j=2}^q \gamma_j^u \sum_{m=0}^j \sum_{l=0}^1 (-1)^{l+m-1} C_1^j C_m^l \frac{r}{2\pi(r^2 - x_{n+k-m} \bar{x}_{n+k-l})} \\
& - h^2 \sum_{j=\delta_{t0}}^k \beta_j \sum_{l=0}^1 (-1)^{l-1} C_1^l \frac{r \bar{x}_{n+k-l}^2}{\pi(r^2 - \bar{x}_{n+k-l} x_{n+k-j})^3}, \quad i = q+1(1)k.
\end{aligned}$$

Hence we get the following:

**Theorem 8 :** The optimal multistep method (15) interpolatory for polynomials of degree  $q$  is characterized by that it is locally interpolatory for functions

$$\{x^i; i = 1(1)q\} \cup \{h_i(x); i = q+1(1)k\},$$

where  $h_i(x) = \sum_{l=0}^1 (-1)^{l-1} C_1^l \frac{1}{(r^2 - x \bar{x}_{n+k-l})}$ ,  $i = q+1(1)k$ .

We implemented  $\beta$ -optimal method interpolatory for polynomials of degree 2, and the numerical results turn out to be the same as those of  $\beta$ -optimal method with restriction (10), as is to be expected theoretically also. The two procedures in this case thus becoming only alternate implementations of theoretically the same formulae. Hence we are presenting the numerical results for  $\beta$ -optimal method interpolatory for polynomials of degree 3 which constrains more than the requirement (10).

From Table 6a, it is seen that the optimal  $\hat{\beta}_{1n}$ 's are depending on the nodal points and in a neighborhood of the origin the optimal  $\hat{\beta}_{1n}$ 's are close to the corresponding usual  $\beta_1$ 's. At a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ , however in a neighborhood of  $x=0$ ,  $\|\hat{T}_n\|^2$  and  $\|T_n\|^2$  are almost equal. We also note that  $\hat{\beta}_{0n} = \hat{\beta}_{2n}$ , and that  $\hat{\beta}_{0n} + \hat{\beta}_{1n} + \hat{\beta}_{2n} = 1$ , which can be shown to be true on theoretical grounds also.

In Table 6b, the numerical results for BVP's with equations 1-9 and 19-24 are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ; and the results for BVP's with equations 10-18 are obtained with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$ . Table 5b reveals that as compared to usual method,  $\beta$ -optimal method interpolatory for polynomials of degree 3 for BVP is two decimal places better on equation 18; one decimal place better on equations 1,3-10,13,14,16,17,20-24; and just better on equations 2,11,12 ,15,19.

Table - 6a

$x_n$	$h^2\hat{\beta}_{0n}$	$h^2\hat{\beta}_{1n}$	$h^2\hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.7339E-03	.8532E-02	.7339E-03	.1000E-01	.1102E-05	.3392E-06
-1.6	.7785E-03	.8443E-02	.7785E-03	.1000E-01	.2478E-07	.8666E-08
-1.5	.7988E-03	.8402E-02	.7988E-03	.1000E-01	.1375E-08	.5247E-09
-1.4	.8098E-03	.8380E-02	.8098E-03	.1000E-01	.1326E-09	.5445E-10
-1.3	.8164E-03	.8367E-02	.8164E-03	.1000E-01	.1860E-10	.8190E-11
-1.2	.8207E-03	.8359E-02	.8207E-03	.1000E-01	.3431E-11	.1618E-11
-1.1	.8237E-03	.8353E-02	.8237E-03	.1000E-01	.7802E-12	.3948E-12
-1.0	.8258E-03	.8348E-02	.8258E-03	.1000E-01	.2094E-12	.1140E-12
-.9	.8274E-03	.8345E-02	.8274E-03	.1000E-01	.6440E-13	.3781E-13
-.8	.8287E-03	.8343E-02	.8287E-03	.1000E-01	.2219E-13	.1410E-13
-.7	.8297E-03	.8341E-02	.8297E-03	.1000E-01	.8485E-14	.5866E-14
-.6	.8305E-03	.8339E-02	.8305E-03	.1000E-01	.3576E-14	.2701E-14
-.5	.8312E-03	.8338E-02	.8312E-03	.1000E-01	.1626E-14	.1335E-14
-.4	.8319E-03	.8336E-02	.8319E-03	.1000E-01	.8270E-15	.7366E-15
-.3	.8324E-03	.8335E-02	.8324E-03	.1000E-01	.4759E-15	.4520E-15
-.2	.8329E-03	.8334E-02	.8329E-03	.1000E-01	.2995E-15	.2951E-15
-.1	.8332E-03	.8334E-02	.8332E-03	.1000E-01	.2026E-15	.2023E-15
.0	.8333E-03	.8333E-02	.8333E-03	.1000E-01	.1766E-15	.1766E-15
.1	.8332E-03	.8334E-02	.8332E-03	.1000E-01	.1319E-15	.1316E-15
.2	.8329E-03	.8334E-02	.8329E-03	.1000E-01	.2288E-15	.2244E-15
.3	.8324E-03	.8335E-02	.8324E-03	.1000E-01	.4405E-15	.4166E-15
.4	.8319E-03	.8336E-02	.8319E-03	.1000E-01	.8094E-15	.7190E-15
.5	.8312E-03	.8338E-02	.8312E-03	.1000E-01	.1608E-14	.1318E-14
.6	.8305E-03	.8339E-02	.8305E-03	.1000E-01	.3523E-14	.2648E-14
.7	.8297E-03	.8341E-02	.8297E-03	.1000E-01	.8555E-14	.5937E-14
.8	.8287E-03	.8343E-02	.8287E-03	.1000E-01	.2230E-13	.1420E-13
.9	.8274E-03	.8345E-02	.8274E-03	.1000E-01	.6440E-13	.3781E-13
1.0	.8258E-03	.8348E-02	.8258E-03	.1000E-01	.2094E-12	.1139E-12
1.1	.8237E-03	.8353E-02	.8237E-03	.1000E-01	.7802E-12	.3947E-12
1.2	.8207E-03	.8359E-02	.8207E-03	.1000E-01	.3431E-11	.1618E-11
1.3	.8164E-03	.8367E-02	.8164E-03	.1000E-01	.1860E-10	.8190E-11
1.4	.8098E-03	.8380E-02	.8098E-03	.1000E-01	.1326E-09	.5445E-10
1.5	.7988E-03	.8402E-02	.7988E-03	.1000E-01	.1375E-08	.5247E-09
1.6	.7785E-03	.8443E-02	.7785E-03	.1000E-01	.2478E-07	.8666E-08
1.7	.7339E-03	.8532E-02	.7339E-03	.1000E-01	.1102E-05	.3392E-06

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals for Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for polynomials of degree 3, at the nodal points, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ;  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ ;  $\gamma_0=1$ ,  $\gamma_1=-1$ ,  $\gamma_2=1/12$ , in  $H^2(C_r)$  space.

Table - 6b

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.237657E-04
2	2	.610703E-04	2	.269937E-04
3	3	.430195E-03	2	.664668E-04
4	2	.133292E-03	2	.290150E-04
5	2	.348263E-03	2	.364906E-04
6	3	.127772E-03	2	.310152E-04
7	2	.313175E-04	2	.758292E-05
8	2	.133523E-03	2	.135525E-04
9	2	.735570E-04	2	.758457E-05
10	1	.170782E-02	1	.462419E-03
11	1	.738872E-03	1	.657043E-03
12	1	.718937E-03	1	.653502E-03
13	2	.298705E-05	2	.846255E-06
14	2	.167071E-05	1	.220927E-06
15	2	.392662E-05	2	.147731E-05
16	2	.236517E-05	2	.643198E-06
17	1	.257920E-07	1	.646324E-08
18	1	.151826E-06	1	.374333E-08
19	3	.509406E-03	2	.104757E-03
20	3	.464907E-03	2	.949003E-04
21	2	.304134E-03	2	.185829E-04
22	2	.208304E-03	2	.117549E-04
23	2	.300455E-03	2	.552848E-04
24	2	.287372E-04	2	.434804E-05

Table for number of iterations for convergence of the solution and the average discretisation error using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method in  $H^2(C_r)$  space, interpolatory for polynomials of degree 3, for 24 BVP-s.

The results in Table 6c for IVP's with equations 1-9 and 19-24 are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$ ; and the results for IVP's with equations 10 - 18 are obtained with  $r=2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$ . Table 6c reveals that as compared to usual method,  $\beta$ -optimal method interpolatory for polynomials of degree 3 for IVP is two decimal places better on equations 21,22; one decimal place better on equations 1,2,4-20,23,24; and just better on equation 3.

Table - 6c

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.159452E-02
2	4	.250457E-02	3	.814122E-03
3	8	.760039E-01	5	.119945E-01
4	5	.132805E-01	4	.141468E-02
5	4	.716823E-02	3	.617527E-03
6	5	.356402E-02	4	.924177E-03
7	4	.354625E-03	3	.935203E-04
8	4	.211573E-02	3	.258687E-03
9	3	.604542E-03	3	.763205E-04
10	4	.119641E-01	4	.738774E-02
11	4	.130464E+00	4	.191610E-01
12	4	.129361E+00	4	.184267E-01
13	2	.128426E-05	2	.370888E-06
14	2	.831124E-06	1	.169616E-06
15	2	.193794E-05	2	.750076E-06
16	2	.130401E-05	2	.429099E-06
17	1	.456442E-08	1	.708386E-09
18	1	.343343E-07	1	.249902E-08
19	11	.103867E+00	6	.179523E-01
20	4	.188589E-02	3	.433292E-03
21	4	.112364E-02	3	.978492E-04
22	5	.115497E-01	4	.884308E-03
23	4	.610397E-02	3	.901793E-03
24	3	.542588E-04	2	.622506E-05

Table for number of iterations for convergence of the solution and the average discretisation error using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method in  $H^2(C_r)$  space, interpolatory for polynomials of degree 3, for 24 IVP-s.

#### 4.5 Optimal Multistep Methods in $H^2(C_r)$ -Space Interpolatory for A Set of Pre assigned Functions.

To obtain the coefficients  $\hat{\beta}_{j,n}^F$ ,  $j = \delta_{t_0}(1)k$  of a  $\beta$ -optimal multistep method

$$(16) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n}^F f_{n+k-j} = 0,$$

in  $H^2(C_r)$  space with prefixed  $\alpha_j$ 's, interpolatory for  $q$  number of



linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$ ; we have to solve a system of linear equations given in the matrix form as

$$\begin{bmatrix} A & F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} b \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where  $A = (A_{ij})_{i=\delta_{t_0}(1)k, j=\delta_{t_0}(1)k}$ , with

$$A_{ij} = \frac{r}{\pi} \frac{2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3} \left[ 1 + \frac{6x_{n+k-j} \bar{x}_{n+k-1}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})} + \frac{6x_{n+k-j}^2 \bar{x}_{n+k-1}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2} \right],$$

$F = (F_{ij})_{i=1(1)q, j=\delta_{t_0}(1)k}$ , with  $F_{ij} = f_i''(x_{n+k-j})$ ,

$$b = h^2 \left( \hat{\beta}_{\delta_{t_0}, n}^F, \hat{\beta}_{\delta_{t_0}+1, n}^F, \dots, \hat{\beta}_{k, n}^F \right)^T, \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T,$$

$$c = (c_{\delta_{t_0}}, c_{\delta_{t_0}+1}, \dots, c_k)^T, \quad \text{with } c_i = - \sum_{j=0}^k \alpha_j \frac{r}{\pi} \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^3},$$

$$f = [\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q]^T, \quad \text{with } \tilde{f}_i = - \sum_{j=0}^k \alpha_j f_i(x_{n+k-j}).$$

This method is characterized as follows:

**Theorem 9:** The  $\beta$ -optimal multistep method (16), in  $H^2(C_r)$  space, interpolatory for linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$ ; is characterized by that it is locally interpolatory for the functions

$$\{f_1, f_2, \dots, f_q\} \cup \{h_i(x); i = \delta_{t_0}(1)k - q\},$$

$$\text{where } h_i(x) = \frac{x^2}{(r^2 - x \bar{x}_{n+k-1})^3} - \sum_{j=k-q+1}^k \bar{g}_{j+q-k, l+1-\delta_{t_0}} \frac{x^2}{(r^2 - x \bar{x}_{n+k-j})^3}$$

and  $G = P^{-1}E = (g_{ij})_{\substack{i=1(1)q \\ j=1(1)k-q-\delta_{t0}+1}},$  where

$$P = \left( f''_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=k-q+1(1)k}} \quad \text{and} \quad E = \left( f''_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=\delta_{t0}(1)k-q}}.$$

Proof: The proof follows from Theorem 8 of Chapter 1.

To obtain the coefficients  $\hat{\alpha}_{j,n}^F$ ,  $j = 0(1)k$  of an  $\alpha$ -optimal multistep method

$$(17) \quad \sum_{j=0}^k \hat{\alpha}_{j,n}^F y_{n+k-j} + h^2 \sum_{j=\delta_{t0}}^k \beta_j f_{n+k-j} = 0,$$

in  $H^2(C_r)$  space with prefixed  $\beta_j$ 's, interpolatory for  $q$  number of linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$ ; we have to solve a system of linear equations given in the matrix form as

$$\begin{bmatrix} A & F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} a \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where  $A = (A_{ij})_{\substack{i=0(1)k \\ j=0(1)k}};$  with  $A_{ij} = \frac{r}{2\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})},$

$F = (F_{ij})_{\substack{i=1(1)q \\ j=0(1)k}};$  with  $F_{ij} = f_i(x_{n+k-j}),$

$a = (\hat{\alpha}_{0,n}^F, \hat{\alpha}_{1,n}^F, \dots, \hat{\alpha}_{k,n}^F)^T, \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T,$

$c = (c_0, c_1, \dots, c_k)^T;$  with  $c_i = -h^2 \sum_{j=\delta_{t0}}^k \beta_j \frac{r}{\pi} \frac{x_{n+k-1}^2}{(r^2 - x_{n+k-1} \bar{x}_{n+k-j})^3},$

$f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T;$  with  $\tilde{f}_i = -h^2 \sum_{j=\delta_{t0}}^k \beta_j f''_i(x_{n+k-j}).$

This method is characterized as follows.

Theorem 10: An  $\alpha$ -optimal multistep method (10), in  $H^2(C_r)$  space interpolatory for arbitrary functions  $f_1, f_2, \dots, f_q$  is characterized by that it is locally interpolatory for the functions

$$\{f_1, f_2, \dots, f_q\} \cup \{h_i(x); i = 0(1)k-q\},$$

$$\text{where } h_i(x) = \frac{1}{(r^2 - x\bar{x}_{n+k-1})} - \sum_{j=k-q+1}^k \bar{g}_{j+q-k, i+1} \frac{1}{(r^2 - x\bar{x}_{n+k-j})},$$

$$\text{and } G = P^{-1}E = (g_{ij})_{\substack{i=1(1)q \\ j=1(1)k-q-\delta_{t0}+1}}, \quad \text{where}$$

$$P = \left( f_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=k-q+1(1)k}} \quad \text{and} \quad E = \left( f_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=0(1)k-q}}.$$

Proof: The proof follows from the Theorem 9 of Chapter 1.

In the following tables we are presenting the numerical results of  $\beta$ -optimal method interpolatory for linearly independent functions  $\exp(1.6x)$  and  $\exp(-1.6x)$ .

Table 7a reveals that the optimal  $\hat{\beta}_{in}$ 's are depending on nodal points and in a neighborhood of the origin the optimal  $\hat{\beta}_{in}$ 's are close to the corresponding usual  $\beta_i$ 's. At a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$  but at  $x = -.1, 0, .1$ ; this inequality does not hold in the numerical values. This must be because of round-offs of order  $O(10^{-17})$ .

The numerical results of Table 7b for BVP's with equations 1-9 and 19-24 are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ; and the results for BVP's with equations 10-18 are obtained with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$ . Table 7b reveals that as compared to usual method  $\beta$ -optimal method interpolatory for linearly independent functions  $\exp(1.6x)$  and  $\exp(-1.6x)$  for BVP's is one

decimal place better on equations 1,3-10, 14,16-18,20-24; and just better on equations 2,11-13,15 and 19.

Table - 7a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.7321E-03	.8538E-02	.7321E-03	.1000E-01	.1102E-05	.3305E-06
-1.6	.7765E-03	.8448E-02	.7765E-03	.1000E-01	.2478E-07	.8311E-08
-1.5	.7969E-03	.8407E-02	.7969E-03	.1000E-01	.1375E-08	.4946E-09
-1.4	.8079E-03	.8385E-02	.8079E-03	.1000E-01	.1326E-09	.5047E-10
-1.3	.8147E-03	.8371E-02	.8147E-03	.1000E-01	.1860E-10	.7485E-11
-1.2	.8191E-03	.8362E-02	.8191E-03	.1000E-01	.3431E-11	.1466E-11
-1.1	.8222E-03	.8356E-02	.8222E-03	.1000E-01	.7802E-12	.3572E-12
-1.0	.8246E-03	.8351E-02	.8246E-03	.1000E-01	.2094E-12	.1039E-12
-.9	.8265E-03	.8347E-02	.8265E-03	.1000E-01	.6440E-13	.3503E-13
-.8	.8280E-03	.8344E-02	.8280E-03	.1000E-01	.2219E-13	.1335E-13
-.7	.8293E-03	.8342E-02	.8293E-03	.1000E-01	.8485E-14	.5682E-14
-.6	.8304E-03	.8339E-02	.8304E-03	.1000E-01	.3576E-14	.2663E-14
-.5	.8312E-03	.8338E-02	.8312E-03	.1000E-01	.1626E-14	.1329E-14
-.4	.8319E-03	.8336E-02	.8319E-03	.1000E-01	.8270E-15	.7351E-15
-.3	.8323E-03	.8335E-02	.8323E-03	.1000E-01	.4759E-15	.4525E-15
-.2	.8326E-03	.8335E-02	.8326E-03	.1000E-01	.2995E-15	.2988E-15
-.1	.8328E-03	.8334E-02	.8328E-03	.1000E-01	.2026E-15	.2090E-15
.0	.8329E-03	.8334E-02	.8329E-03	.1000E-01	.1766E-15	.1846E-15
.1	.8328E-03	.8334E-02	.8328E-03	.1000E-01	.1319E-15	.1383E-15
.2	.8326E-03	.8335E-02	.8326E-03	.1000E-01	.2288E-15	.2281E-15
.3	.8323E-03	.8335E-02	.8323E-03	.1000E-01	.4405E-15	.4172E-15
.4	.8319E-03	.8336E-02	.8319E-03	.1000E-01	.8094E-15	.7174E-15
.5	.8312E-03	.8338E-02	.8312E-03	.1000E-01	.1608E-14	.1311E-14
.6	.8304E-03	.8339E-02	.8304E-03	.1000E-01	.3523E-14	.2610E-14
.7	.8293E-03	.8342E-02	.8293E-03	.1000E-01	.8555E-14	.5753E-14
.8	.8280E-03	.8344E-02	.8280E-03	.1000E-01	.2230E-13	.1345E-13
.9	.8265E-03	.8347E-02	.8265E-03	.1000E-01	.6440E-13	.3503E-13
1.0	.8246E-03	.8351E-02	.8246E-03	.1000E-01	.2094E-12	.1039E-12
1.1	.8222E-03	.8356E-02	.8222E-03	.1000E-01	.7802E-12	.3571E-12
1.2	.8191E-03	.8362E-02	.8191E-03	.1000E-01	.3431E-11	.1466E-11
1.3	.8147E-03	.8371E-02	.8147E-03	.1000E-01	.1860E-10	.7485E-11
1.4	.8079E-03	.8385E-02	.8079E-03	.1000E-01	.1326E-09	.5047E-10
1.5	.7969E-03	.8407E-02	.7969E-03	.1000E-01	.1375E-08	.4946E-09
1.6	.7765E-03	.8448E-02	.7765E-03	.1000E-01	.2478E-07	.8311E-08
1.7	.7321E-03	.8538E-02	.7321E-03	.1000E-01	.1102E-05	.3305E-06

Table for optimal  $\beta$ 's, their row-sum and the square of the norm of local truncation error functionals in  $H^2(C_r)$ -space for Cowell's usual 3-point method and the corresponding  $\beta$ -optimal method, interpolatory for  $\exp(\pm 1.6x)$  at the nodal points with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_3=-1$ ;  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ .

Table - 7b

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.257124E-04
2	2	.610703E-04	2	.236998E-04
3	3	.430195E-03	2	.668193E-04
4	2	.133292E-03	2	.263115E-04
5	2	.348263E-03	2	.241046E-04
6	3	.127772E-03	2	.307981E-04
7	2	.313175E-04	2	.751404E-05
8	2	.133523E-03	2	.145138E-04
9	2	.735570E-04	2	.795490E-05
10	1	.170782E-02	1	.386341E-03
11	1	.738872E-03	1	.511681E-03
12	1	.718937E-03	1	.507192E-03
13	2	.298705E-05	2	.105352E-05
14	2	.167071E-05	2	.459325E-06
15	2	.392662E-05	2	.154223E-05
16	2	.236517E-05	2	.756348E-06
17	1	.257920E-07	1	.708103E-08
18	1	.151826E-06	1	.167395E-07
19	3	.509406E-03	2	.106010E-03
20	3	.464907E-03	2	.961642E-04
21	2	.304135E-03	2	.232490E-04
22	2	.208304E-03	2	.149971E-04
23	2	.300455E-03	2	.438887E-04
24	2	.287372E-04	2	.613905E-05

Table for number of iterations for convergence of the solution and the average discretisation error using Cowell's 3-points usual and the corresponding  $\beta$ -optimal method in  $H^2(C_r)$  space, interpolatory for functions  $\exp(\pm 1.6x)$ , for 24 BVP-s.

The results of Table 7c are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$  for IVP's with equations 1-9 and 19-24, and with  $r=2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$  for IVP's with equations 10-18. Table 7c reveals that as compared to usual method  $\beta$ -optimal method interpolatory for linearly independent functions  $\exp(\pm 1.6x)$  for IVP is one decimal place better on equations 1,2,4-13 16-24; and just better on equations 3,14 and 15.

Table - 7c

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.172832E-02
2	4	.250457E-02	3	.676315E-03
3	8	.760039E-01	5	.120006E-01
4	5	.132805E-01	4	.116489E-02
5	4	.716823E-02	3	.301960E-03
6	5	.356402E-02	4	.915038E-03
7	4	.354625E-03	3	.925088E-04
8	4	.211573E-02	3	.278186E-03
9	3	.604542E-03	3	.806865E-04
10	4	.119641E-01	4	.725326E-02
11	4	.130464E+00	4	.269349E-01
12	4	.129361E+00	4	.262629E-01
13	2	.128426E-05	2	.720024E-06
14	2	.831124E-06	2	.469907E-06
15	2	.193794E-05	2	.104693E-05
16	2	.130401E-05	2	.683572E-06
17	1	.456442E-08	1	.891521E-09
18	1	.343343E-07	1	.242165E-08
19	11	.103867E+00	6	.180424E-01
20	4	.188589E-02	3	.436142E-03
21	4	.112364E-02	3	.117028E-03
22	5	.115497E-01	4	.108422E-02
23	4	.610397E-02	3	.631845E-03
24	3	.542588E-04	2	.955502E-05

Table for number of iterations for convergence of the solution and the average discretisation error for using Cowell's usual method with function evaluation at three points and the corresponding  $\beta$ -optimal method in  $H^2(C_r)$ -space, interpolatory for linearly independent functions  $\exp(\pm 1.6x)$  applied on 24 IVP-s.

In Figures to follow, we are using the following notations.

Beta-Optimal(1) :  $\beta$ -optimal method

Beta-Optimal(2) :  $\beta$ -optimal method with restriction

Beta-Optimal(3) :  $\beta$ -optimal method interpolatory for polynomials of degree 3.

Beta-Optimal(4) :  $\beta$ -optimal method interpolatory for  $\exp(\pm 1.6x)$ .

All the above beta-optimal methods are corresponding to Cowell's usual method with function evaluation at three points.

Beta-Optimal(3) :  $\beta$ -optimal method interpolatory for polynomials of degree 3.

Beta-Optimal(4) :  $\beta$ -optimal method interpolatory for  $\exp(\pm 1.6x)$ .

All the above beta-optimal methods are corresponding to Cowell's usual method with function evaluation at three points.

Beta-Optimal(5) :  $\beta$ -optimal method corresponding to Störmer's usual method with function evaluation at five points.

The following graphs are drawn according to the corresponding tabular results given earlier. Figure 1 is drawn from the tabular results given in Tables 2a, 5a, 6a and 7a. Figure 2 is drawn from the tabular results given in Table 3a. Figure 3 is drawn from the tabular results given in Table 4a. Figure 1 shows that the  $\beta$ -optimal method is giving more optimized results than other optimal methods. From figure 3, we conclude that  $\alpha$ -optimal method is not giving so promising results, and even though the numerical calculations are done in double precision, near the origin, the square norm of local truncation errors being of the order of  $10^{-16}$  or  $10^{-17}$ , these results are not very reliable.

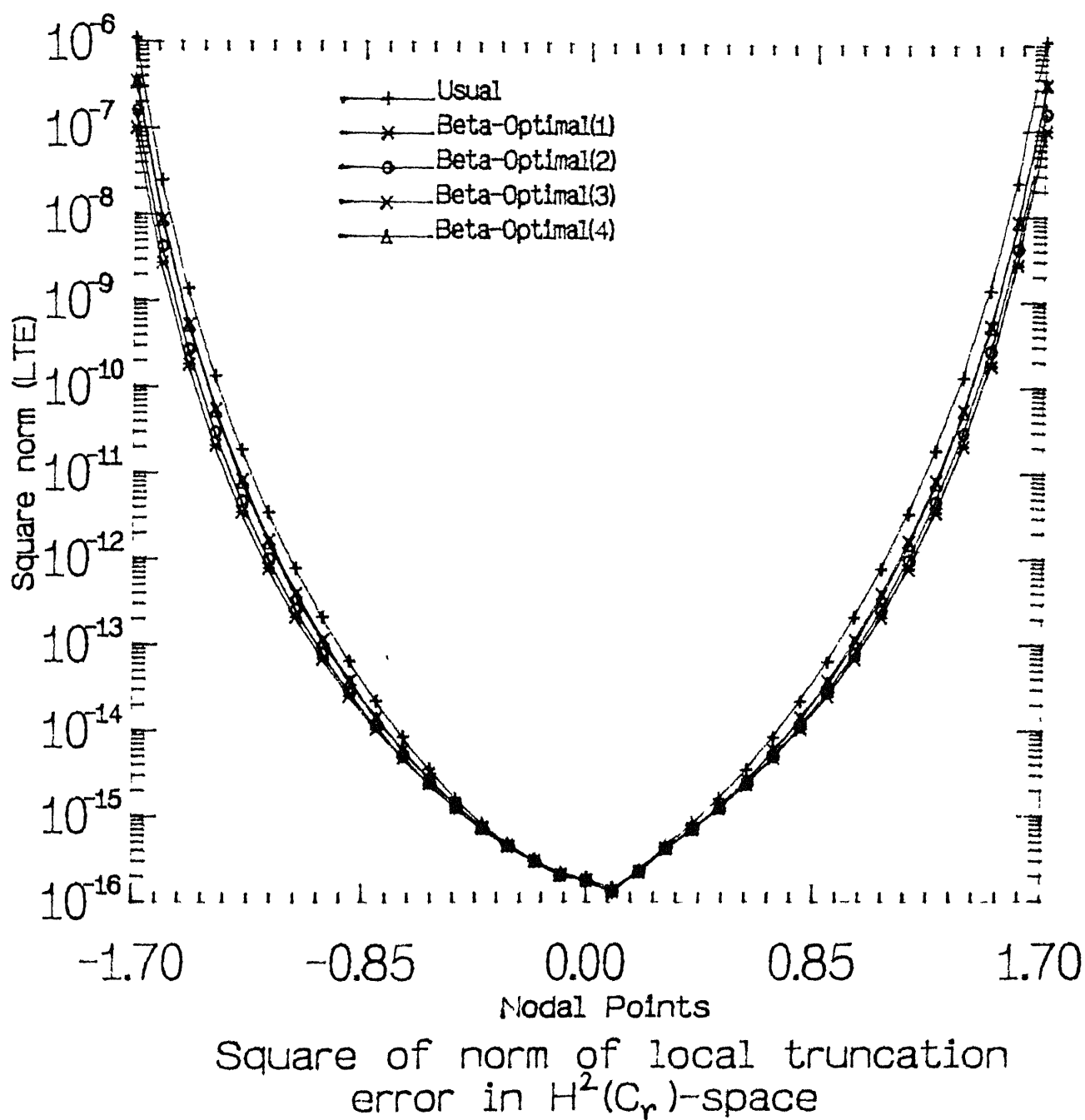


FIG. 1



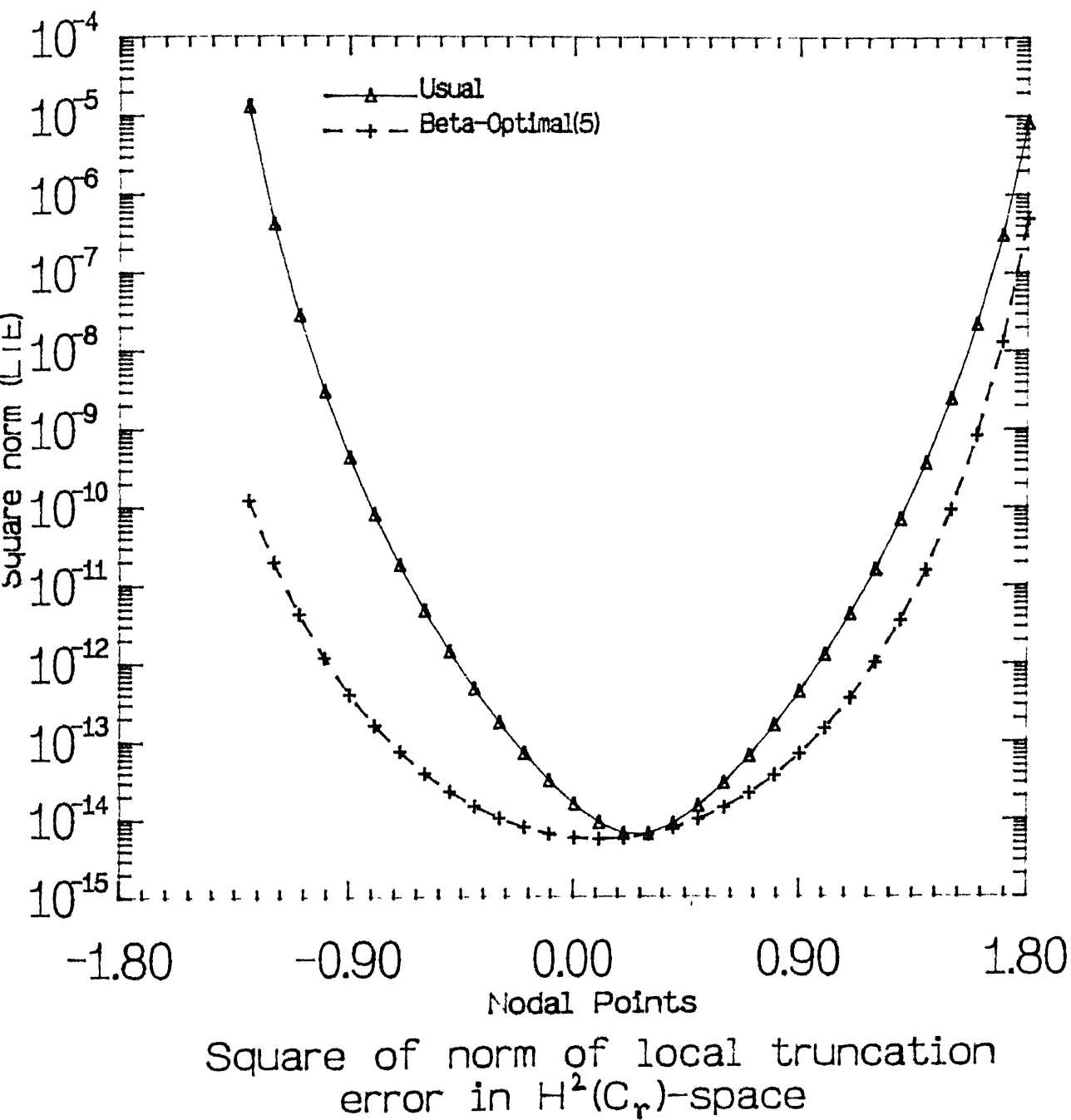


FIG. 2

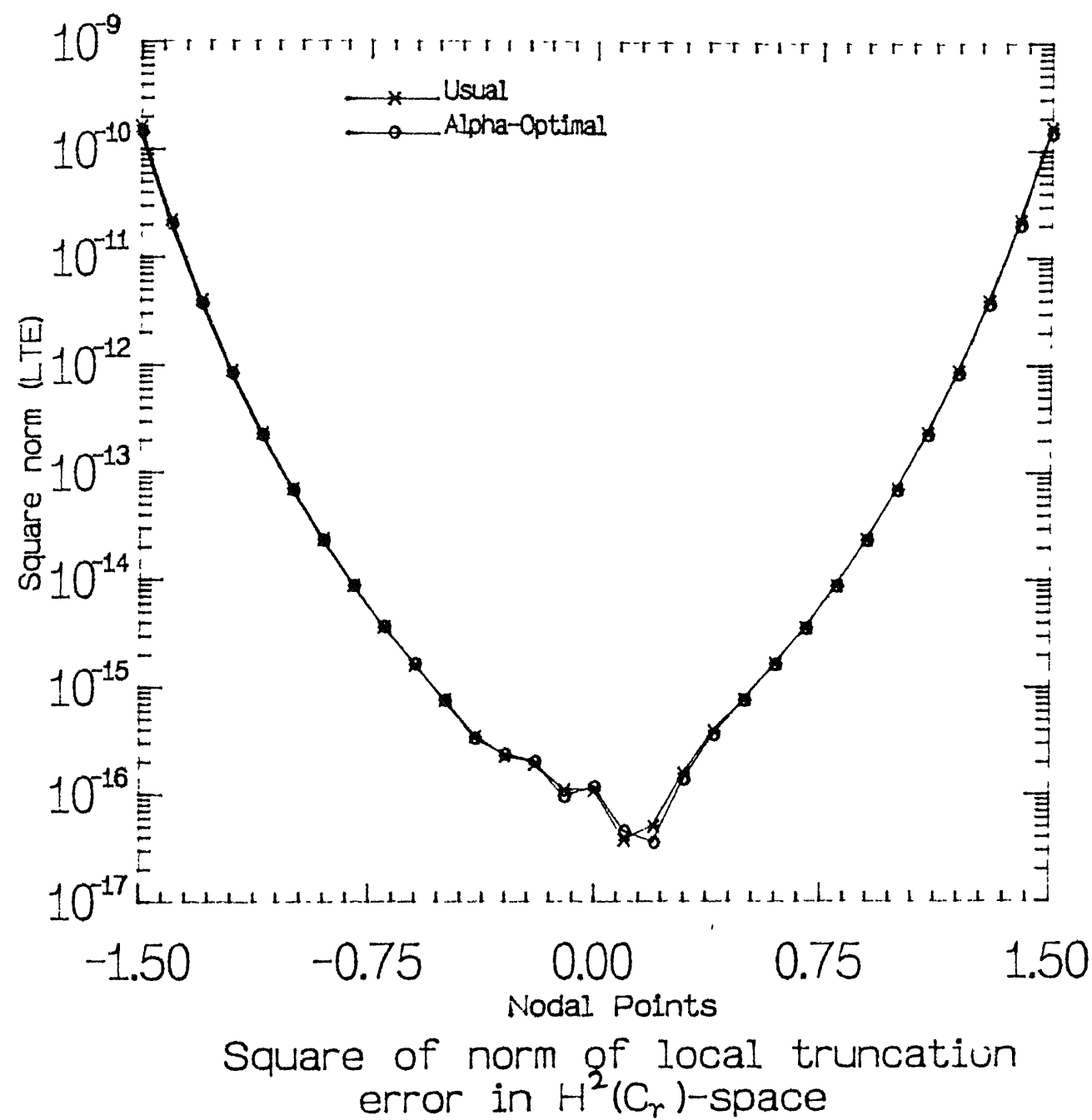


FIG. 3

#### 4.6 Behavior of the Coefficients As $r \rightarrow \infty$

From Tables we observe that the coefficients of  $\beta$ -optimal multistep methods,  $\beta$ -optimal multistep methods interpolatory for polynomials of certain degree or interpolatory for linearly independent arbitrary functions are approximately the same as the corresponding coefficients of the usual method at the points under consideration in a neighborhood of the center of the circle  $C_r$ , which in other words are the points that are well inside the domain of the space. Such a kind of property of the coefficients has been discussed in [16], [82], [113], [154] and [158]. For the space  $H^2(C_r)$ , the behavior of the coefficients in the quadrature formulae as  $r \rightarrow \infty$  is studied by Barnhill [16], Kaul [113] and Finmay and Price Jr. [82] and a similar study for the optimal multistep methods for first order differential equations is done by Brij Bhusan [27]. Here we obtain the corresponding results for the optimal multistep methods in  $H^2(C_r)$  space for the second order differential equations.

Theorem 11: Let

$$(18) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \beta_j^F f_{n+k-j} = 0$$

be a multistep method interpolatory for functions  $\{\phi_{\delta_{t_0}}, \dots, \phi_k\}$  and let the matrix  $[\phi_1''(x_{n+k-j})]_{1,j=\delta_{t_0}(1)k}$  be nonsingular. Let

$$(19) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^F f_{n+k-j} = 0$$

be an optimal multistep formula in  $H^2(C_r)$  space, interpolatory for

functions  $\{\phi_{\delta_{t_0}}, \dots, \phi_{p-1+\delta_{t_0}}\}$ ,  $0 \leq p \leq k+\delta_{t_1} = k+1-\delta_{t_0}$ , ( $p = 0$  implies non-interpolatory case for these functions). If the local truncation error functional  $\hat{T}_{nr}^F$  for (19) satisfies

$$(20) \quad \lim_{r \rightarrow \infty} \hat{T}_{nr}^F(\phi_j) = 0, \quad j = p+\delta_{t_0}(1)k,$$

then

$$(21) \quad \lim_{r \rightarrow \infty} \hat{\beta}_{j,r}^F = \beta_j^F, \quad j = \delta_{t_0}(1)k.$$

Proof: For the errors for the functions  $\phi_{\delta_{t_0}}, \phi_{\delta_{t_0+1}}, \dots, \phi_k$  in (18) and (19) we get

$$(22) \quad \sum_{j=0}^k \alpha_j \phi_1(x_{n+k-j}) - h^2 \sum_{j=\delta_{t_0}}^k \beta_j^F \phi_1''(x_{n+k-j}) = 0, \quad i = \delta_{t_0}(1)k,$$

and

$$(23) \quad \sum_{j=0}^k \alpha_j \phi_i(x_{n+k-j}) - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^F \phi_i''(x_{n+k-j}) = \hat{T}_{nr}^F(\phi_i), \quad i = \delta_{t_0}(1)k.$$

Subtracting (23) from (22), we get

$$- h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^F \phi_i''(x_{n+k-j}) + h^2 \sum_{j=\delta_{t_0}}^k \beta_j^F \phi_i''(x_{n+k-j}) = \hat{T}_{nr}^F(\phi_i), \quad i = \delta_{t_0}(1)k.$$

In matrix notation, we have

$$(24) \quad h^2 G \hat{B}_r^F - h^2 G B^F + \hat{T}_{nr}^F = 0,$$

where

$$G = [\phi_1''(x_{n+k-j})]_{i,j=\delta_{t_0}(1)k},$$

$$\hat{B}_r^F = [\hat{\beta}_{\delta_{t_0},r}^F, \hat{\beta}_{\delta_{t_0+1},r}^F, \dots, \hat{\beta}_{k,r}^F]^T,$$

$$B^F = [\beta_{\delta_{t_0}}^F, \beta_{\delta_{t_0+1}}^F, \dots, \beta_k^F]^T,$$

and 
$$\hat{T}_{nr}^F = \left[ \hat{T}_{nr}^F(\phi_{\delta_{t_0}}), \hat{T}_{nr}^F(\phi_{\delta_{t_0}+1}), \dots, \hat{T}_{nr}^F(\phi_k) \right]^T.$$

since the optimal method (19) is interpolatory for  $\phi_{\delta_{t_0}}, \dots, \phi_{p-1+\delta_{t_0}}$

$$\hat{T}_{nr}^F(\phi_{\delta_{t_0}}) = \dots = \hat{T}_{nr}^F(\phi_{p+\delta_{t_0}-1}) = 0$$

by (20),  $\lim_{r \rightarrow \infty} \hat{T}_{nr}^F = 0$ . Since  $G$  is nonsingular, we get from (24)

$$\lim_{r \rightarrow \infty} \hat{B}_r^F = \lim_{r \rightarrow \infty} B^F$$

or, 
$$\lim_{r \rightarrow \infty} \hat{\beta}_{j,r}^F = \beta_j^F, \quad j = \delta_{t_0}(1)k. \quad \text{Hence the proof.}$$

Theorem 12: Let  $\alpha_j$ ,  $j = 0(1)k$  be constants such that  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j\alpha_j = 0$ . Then, the  $H^2(C_r)$ -coefficients  $\hat{\beta}_{j,r}$  of an optimal multistep method

$$(25) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^F f_{n+k-j} = 0,$$

approach the coefficients  $\beta_j$  of the usual multistep method

$$(26) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0$$

of maximal polynomial precision.

Proof: Since the usual method (26) is interpolatory for polynomials of maximal degree and it is assumed that  $\alpha_j$ 's are known constants with  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j\alpha_j = 0$ , the coefficients  $\beta_j$ 's of the usual method (26) can uniquely be determined by using that it is interpolatory for polynomials  $x^2, x^3, \dots, x^{k+\delta_{t_1}+1}$ . In view of Theorem 1, we need to prove that the local truncation error functional  $\hat{T}_{nr}$  for the optimal method (25) satisfies

$$\lim_{r \rightarrow \infty} \hat{T}_{nr} (x^i) = 0, \quad 0 \leq i \leq k + \delta_{t_1} + 1$$

If  $T_n$  is the local truncation error functional for the usual method then in the space of optimization, e.g. in  $H^2(C_r)$  space we have

$$\|\hat{T}_{nr}\| \leq \|T_n\|.$$

Since  $\{\psi_k\}_{k=0}^{\infty}$  is a complete orthonormal sequence in  $H^2(C_r)$  space, where  $\psi_k(z) = \frac{z^k}{\sqrt{2\pi r} r^k}$ ,  $k \geq 0$ ; and  $T_n(x^j) = 0$ ,  $0 \leq j \leq k + \delta_{t_1} + 1$ ,

$$\sum_{i=0}^{\infty} |\hat{T}_{nr}(\psi_i)|^2 \leq \sum_{j=0}^{\infty} |T_n(\psi_j)|^2 = \sum_{j=k+\delta_{t_1}+2}^{\infty} \frac{1}{2\pi} r^{-(2j+1)} |T_n(x^j)|^2$$

$$\text{or, } \sum_{i=0}^{\infty} \frac{1}{2\pi} r^{-(2i+1)} |\hat{T}_{nr}(x^i)|^2 \leq \sum_{j=k+\delta_{t_1}+2}^{\infty} \frac{1}{2\pi} r^{-(2j+1)} |T_n(x^j)|^2$$

Thus for each fixed  $i$ ,  $0 \leq i \leq k + \delta_{t_1} + 1$ ,

$$|\hat{T}_{nr}(x^i)|^2 \leq \sum_{j=k+\delta_{t_1}+2}^{\infty} r^{-2(j-1)} |T_n(x^j)|^2.$$

The expression on the right hand side approaches zero as  $r \rightarrow \infty$ .

$$\text{So, } \lim_{r \rightarrow \infty} \hat{T}_{nr} (x^i) = 0, \quad 0 \leq i \leq k + \delta_{t_1} + 1.$$

Hence the result.

Theorem 13: Let  $\alpha_j$ ,  $j = 0(1)k$  be constants such that  $\sum_{j=0}^k \alpha_j = 0$ ,

and  $\sum_{j=1}^k j\alpha_j = 0$ . Then the  $H^2(C_r)$ -coefficients  $\hat{\beta}_{j,r}^p$  of the optimal multistep method

$$(27) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^p f_{n+k-j} = 0,$$

interpolatory for polynomials of degree  $p \leq k + \delta_{t_1} + 1$ , approach as  $r \rightarrow \infty$ , the coefficients  $\beta_j$  of the corresponding multistep method of highest degree polynomial precision

$$\sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0.$$

Proof: Since  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j\alpha_j = 0$ , the method (27) is interpolatory for constants and  $x$ . As it is interpolatory for polynomials of degree  $p$ , the local truncation error functional  $\hat{T}_{nr}^p$  of (27) satisfies

$$\hat{T}_{nr}^p(x^i) = 0, \quad 0 \leq i \leq p.$$

In view of Theorem 11, it is sufficient to prove that

$$\lim_{r \rightarrow \infty} \hat{T}_{nr}^p(x^i) = 0, \quad p+1 \leq i \leq k + \delta_{t_1} + 1.$$

If  $T_n$  denotes the local truncation error functional of the usual method, then in the space of optimization viz. in  $H^2(C_r)$ -space

$$\|\hat{T}_{nr}^p\| \leq \|T_n\|.$$

As given in Theorem 12,

$$\sum_{i=0}^{\infty} |\hat{T}_{nr}^p(\psi_i)|^2 \leq \sum_{j=0}^{\infty} |T_n(\psi_j)|^2 = \sum_{j=k+\delta_{t_1}+2}^{\infty} \frac{1}{2\pi} r^{-(2j+1)} |T_n(x^j)|^2.$$

Since  $\hat{T}_{nr}^p(x^i) = 0$ , for  $0 \leq i \leq p$ , for  $i \geq p+1$ , we get

$$|\hat{T}_{nr}^p(x^i)|^2 \leq \sum_{j=k+\delta_{t_1}+2}^{\infty} r^{-2(j-1)} |T_n(x^j)|^2.$$

The expression on the right approaches zero as  $r \rightarrow \infty$ , for  $i \leq k + \delta_{t_1} + 1$ .

Hence

$$\lim_{r \rightarrow \infty} \hat{T}_{nr} (x^1) = 0, \quad p+1 \leq 1 \leq k+\delta_{t_1}+1$$

Hence the result.

Theorem 14: Let  $\alpha_j$ ,  $j = 0(1)k$  be constants such that  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j\alpha_j = 0$ . Let the  $(k+\delta_{t_1}) \times (k+\delta_{t_1})$  matrix  $M$  be nonsingular, where

$$M = \begin{bmatrix} \phi_1''(x_{n+k-1+\delta_{t_1}}) & \dots & \phi_1''(x_n) \\ \dots & \dots & \dots \\ \phi_p''(x_{n+k-1+\delta_{t_1}}) & \dots & \phi_p''(x_n) \\ 1 & \dots & 1 \\ x_{n+k-1+\delta_{t_1}} & \dots & x_n \\ \dots & \dots & \dots \\ x_{n+k-1+\delta_{t_1}}^{k-p-\delta_{t_0}} & \dots & x_n^{k-p-\delta_{t_0}} \end{bmatrix}.$$

Then the coefficients  $\hat{\beta}_{k-j,r}^F$ 's in  $H^2(C_r)$  space of the optimal multistep method

$$(28) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^F f_{n+k-j} = 0,$$

interpolatory for functions  $\{\phi_1, \phi_2, \dots, \phi_p\}$ ,  $p \leq k+\delta_{t_1}$  approach the coefficients  $\beta_j^F$ 's of the unique multistep method

$$(29) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \beta_j^F f_{n+k-j} = 0,$$

which is interpolatory for functions

$$\{\phi_1, \phi_2, \dots, \phi_p, x^2, x^3, \dots, x^{k+\delta_{t_1}+1-p}\}.$$

Proof: As the matrix  $M$  is non-singular, for given  $\alpha_j$ 's,  $\beta_j^F$ 's of the method (29) can be determined uniquely. Since both the methods



are interpolatory for functions  $\{\phi_1, \phi_2, \dots, \phi_p\}$ , by Theorem 11, it is sufficient to prove that the local truncation error functional  $\hat{T}_{nr}^F$  of (28) satisfies

$$\lim_{r \rightarrow \infty} \hat{T}_{nr}^F(x^i) = 0, \quad 2 \leq i \leq k + \delta_{t_1} + 1 - p.$$

Since  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j \alpha_j = 0$ , the usual method (29) is exact for constants and  $x$ . Also it is exact for  $x^2, x^3, \dots, x^{k + \delta_{t_1} + 1 - p}$ . If  $T_n^F$  is the local truncation error functional of the usual method (29), then  $T_n^F(x^i) = 0$ , for  $0 \leq i \leq k + \delta_{t_1} + 1 - p$ . As in Theorem 12,

$$\sum_{i=0}^{\infty} |\hat{T}_{nr}^F(\psi_i)|^2 \leq \sum_{j=0}^{\infty} |T_n^F(\psi_j)|^2 = \sum_{j=k + \delta_{t_1} + 2 - p}^{\infty} \frac{1}{2\pi} r^{-(2j+1)} |T_n^F(x^j)|^2$$

$$\text{or, } \sum_{i=0}^{\infty} \frac{1}{2\pi} r^{-(2i+1)} |\hat{T}_{nr}^F(x^i)|^2 \leq \sum_{j=k + \delta_{t_1} + 2 - p}^{\infty} \frac{1}{2\pi} r^{-(2j+1)} |T_n^F(x^j)|^2.$$

Thus for each fixed  $i$ ,  $i \geq 0$

$$|\hat{T}_{nr}^F(x^i)|^2 \leq \sum_{j=k + \delta_{t_1} + 2 - p}^{\infty} r^{-2(j-1)} |T_n^F(x^j)|^2.$$

For  $2 \leq i \leq k + \delta_{t_1} + 1 - p$ , the expression on the right hand side approaches zero as  $r \rightarrow \infty$ . Hence

$$\lim_{r \rightarrow \infty} \hat{T}_{nr}^F(x^i) = 0, \quad 2 \leq i \leq k + \delta_{t_1} + 1 - p.$$

Hence the result.

#### 4.7 An Asymptotic Estimate of Local Truncation Error

In this section, we shall give asymptotic estimate of the optimal  $\beta$ -coefficients and the local truncation error functionals

in  $H^2(C_r)$  space, for both the usual and the  $\beta$ -optimal methods and compare them in some simple cases. First, we shall consider Störmer's method with function evaluation at one point. Next, we shall consider Cowell's method with function evaluation at three points as the usual method. For solving numerically a differential equation, with some initial or boundary conditions, of the form

$$(I) \quad y'' = f(x, y),$$

Störmer's usual method with function evaluation at one point is

$$(30) \quad \sum_{i=0}^2 \alpha_i y_{n-1+i} + h^2 f_n = 0,$$

where  $\alpha_0 = -1$ ,  $\alpha_1 = 2$ ,  $\alpha_2 = -1$ ; the corresponding  $\beta$ -optimal method is

$$(31) \quad \sum_{i=0}^2 \alpha_i y_{n-1+i} + h^2 \hat{\beta}_{1,n} f_n = 0.$$

The normal equation for obtaining  $\hat{\beta}_{1,n}$  is given by

$$\sum_{j=0}^2 \alpha_j D2(x_{n-1+j}, \bar{x}_n) + h^2 \hat{\beta}_{1,n} D2''(x_n, \bar{x}_n) = 0,$$

$$\text{so that} \quad \hat{\beta}_{1,n} = - \frac{1}{h^2} \frac{\sum_{j=0}^2 \alpha_j D2(x_{n-1+j}, \bar{x}_n)}{D2''(x_n, \bar{x}_n)}.$$

Using the definitions of  $D2(x_{n-1+j}, \bar{x}_n)$  and  $D2''(x_n, \bar{x}_n)$  from (5) and (6), as in the Case 1 of Lemma 5 of Chapter 3,

$$(32) \quad \hat{\beta}_{1,n} = C \left[ \frac{(x-h)^2 (r^2 - x^2)^3}{(r^2 - x^2 + xh)^3} + \frac{(x+h)^2 (r^2 - x^2)^3}{(r^2 - x^2 - xh)^3} - 2x^2 \right]$$

with  $x_n = x$ , and  $C = \frac{(r^2 - x^2)^2}{2h^2(r^4 + 4r^2x^2 + x^4)}$ . Hence

$$\hat{\beta}_{1,n} = C \left[ (x-h)^2 \left\{ 1 + \frac{hx}{r^2 - x^2} \right\}^{-3} + (x+h)^2 \left\{ 1 - \frac{hx}{r^2 - x^2} \right\}^{-3} - 2x^2 \right]$$

$$= 2h^2 c \left[ 1 + \frac{6x^2}{r^2 - x^2} + \frac{6x^2(x^2 + h^2)}{(r^2 - x^2)^2} + \frac{20h^2 x^4}{(r^2 - x^2)^3} + \frac{15h^2 x^4(x^2 + h^2)}{(r^2 - x^2)^4} \right. \\ \left. + \frac{42h^4 x^6}{(r^2 - x^2)^5} + \dots \right].$$

It follows that

$$(33) \quad \hat{\beta}_{1,n} = 1 + \frac{(r^2 - x^2)^2}{(r^4 + 4r^2 x^2 + x^4)} \left\{ \frac{6x^2 h^2}{(r^2 - x^2)^2} + \frac{20x^4 h^2}{(r^2 - x^2)^3} + \frac{15x^6 h^2}{(r^2 - x^2)^4} \right. \\ + \frac{15x^4 h^4}{(r^2 - x^2)^4} + \frac{42x^6 h^4}{(r^2 - x^2)^5} + \frac{28x^8 h^4}{(r^2 - x^2)^6} + \frac{28x^6 h^6}{(r^2 - x^2)^6} + \frac{72x^8 h^6}{(r^2 - x^2)^7} + \frac{45x^{10} h^6}{(r^2 - x^2)^8} + \\ \left. \frac{45x^8 h^8}{(r^2 - x^2)^8} + \frac{110x^{10} h^8}{(r^2 - x^2)^9} + \frac{66x^{12} h^8}{(r^2 - x^2)^{10}} + \dots \right\}.$$

We see that  $\hat{\beta}_{1,n} - 1$  is precisely of order  $h^2$  at a fixed  $x \neq 0$ ; at  $x = 0$ ,  $\hat{\beta}_{1,n} = 1$ , and if  $x = O(h)$ , then  $\hat{\beta}_{1,n} - 1 = O(h^4)$ . Square norm of local truncation error functional for optimal method (31), is given by

$$\|\hat{T}_n\|^2 = \sum_{i=0}^2 \bar{\alpha}_i \left( \sum_{j=0}^2 \alpha_j K(x_{n-1+j}, \bar{x}_{n-1+i}) + h^2 \hat{\beta}_{1,n} \overline{D2(x_{n-1+i}, \bar{x}_n)} \right).$$

As  $\alpha_i$ 's and the nodal points are real, putting  $x_n = x$ , and using values of  $K(x_{n-1+j}, \bar{x}_{n-1+i})$ ,  $D2(x_{n-1+i}, \bar{x}_n)$  from (4) and (5), we get

$$\|\hat{T}_n\|^2 = \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \alpha_j \frac{r}{2\pi(r^2 - x_{n-1+j} x_{n-1+i})} + h^2 \hat{\beta}_{1,n} \sum_{i=0}^2 \alpha_i \frac{r x_{n-1+i}^2}{\pi(r^2 - x_{n-1+i} x_n)^3}.$$

Writing  $x_n = x$ , and substituting the values of  $\alpha_j$ 's, we get

$$(34) \quad \|\hat{T}_n\|^2 = \frac{r}{\pi} \sum_{i=0}^2 \alpha_i \left[ \frac{-1}{2(r^2 - (x-h)(x-(1-i)h))} + \frac{2}{2(r^2 - x(x-(1-i)h))} \right. \\ \left. + \frac{-1}{2(r^2 - (x+h)(x-(1-i)h))} \right] + \frac{r}{\pi} h^2 \hat{\beta}_{1,n} \sum_{i=0}^2 \alpha_i \frac{(x-(1-i)h)^2}{(r^2 - x(x-(1-i)h))^3}.$$

Square norm of local truncation error functional for usual method

(30), is given by

$$\begin{aligned} \|T_n\|^2 = & \sum_{i=0}^2 \bar{\alpha}_i \left( \sum_{j=0}^2 \alpha_j K(x_{n-1+j}, \bar{x}_{n-1+i}) + h^2 \overline{D2}(x_{n-1+i}, \bar{x}_n) \right) \\ & + h^2 \left( \sum_{j=0}^2 \alpha_j D2(x_{n-1+j}, \bar{x}_n) + h^2 D2''(x_n, \bar{x}_n) \right). \end{aligned}$$

Putting  $x_n = x$  and using values of  $K(x_{n-1+j}, \bar{x}_{n-1+i})$ ,  $D2(x_{n-1+i}, \bar{x}_n)$  from (4) and (5), we get

$$\begin{aligned} (35) \quad \|T_n\|^2 = & \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \alpha_j \frac{r}{2\pi[r^2 - (x - (1-j)h)(x - (1-i)h)]} \\ & + h^2 \sum_{i=0}^2 \alpha_i \frac{r(x - (1-i)h)^2}{\pi[r^2 - (x - (1-i)h)x]^3} + h^2 \sum_{j=0}^2 \alpha_j \frac{r(x - (1-j)h)^2}{\pi[r^2 - (x - (1-j)h)x]^3} \\ & + h^4 \frac{2r}{\pi(r^2 - x^2)^5} [r^2(r^2 + 4x^2) + x^4]. \end{aligned}$$

(i) At  $x = 0$ :

From (32),  $\hat{\beta}_{1,n} \Big|_{x=0} = \frac{1}{2h^2} \frac{2h^2}{r^6} \frac{r^{10}}{r^4} = 1$ . From (34) and (35),

$$\|\hat{T}_n\|^2 \Big|_{x=0} = \frac{r}{\pi} \sum_{i=0}^2 \alpha_i \left\{ \left( \frac{-1}{2(r^2 - h^2(1-i))} + \frac{2}{2r^2} + \frac{-1}{2(r^2 + h^2(1-i))} \right) + \frac{h^4}{r^6} (1-i)^2 \right\},$$

and

$$\begin{aligned} \|T_n\|^2 \Big|_{x=0} &= \frac{r}{\pi} \sum_{i=0}^2 \alpha_i \left\{ \left( \frac{-1}{2(r^2 - h^2(1-i))} + \frac{2}{2r^2} + \frac{-1}{2(r^2 + h^2(1-i))} \right) + \frac{h^4}{r^6} (1-i)^2 \right\} \\ &\quad + \frac{r}{\pi} \frac{h^4}{r^6} \sum_{j=0}^2 \alpha_j (1-j)^2 + \frac{r}{\pi} \frac{2h^4}{r^6} \\ &= \|\hat{T}_n\|^2 \Big|_{x=0} - \frac{r}{\pi} \frac{2h^4}{r^6} + \frac{r}{\pi} \frac{2h^4}{r^6}. \end{aligned}$$

Hence,  $\|T_n\|^2 \Big|_{x=0} = \|\hat{T}_n\|^2 \Big|_{x=0}$ , which is otherwise also clear as  $\hat{\beta}_{1n} = 1$ .

(ii) At  $x = h$ :

From (32), we get

$$(36) \quad \hat{\beta}_{1,n} \Big|_{x=h} = \frac{(r^2 - h^2)^2 \cdot (r^6 - 6r^2h^4 + 6h^6)}{(r^2 - 2h^2)^3 \cdot (r^4 + 4r^2h^2 + h^4)}.$$

From (34),

$$\begin{aligned} \|\hat{T}_n\|^2 \Big|_{x=h} &= \frac{r}{\pi} \sum_{i=0}^2 \alpha_i \left[ \left\{ -\frac{1}{2r^2} + \frac{1}{(r^2 - ih^2)} - \frac{1}{2(r^2 - 2ih^2)} \right\} + \hat{\beta}_{1,n} \frac{i^2 h^4}{(r^2 - ih^2)^3} \right] \\ &= \frac{r}{\pi} (A + B), \text{ say.} \end{aligned}$$

Let

$$A_1 = \sum_{i=0}^2 \alpha_i \left( -\frac{1}{2r^2} \right) = 0,$$

$$A_2 = \sum_{i=0}^2 \alpha_i \frac{1}{(r^2 - ih^2)} = -\frac{1}{r^2} + \frac{2}{(r^2 - h^2)} - \frac{1}{(r^2 - 2h^2)},$$

$$\text{and } A_3 = \sum_{i=0}^2 \alpha_i \frac{-1}{2(r^2 - 2ih^2)} = \frac{1}{2r^2} - \frac{1}{(r^2 - 2h^2)} + \frac{1}{2(r^2 - 4h^2)}.$$

$$\begin{aligned} \text{Then } A &= A_1 + A_2 + A_3 = \frac{2}{(r^2 - h^2)} - \frac{2}{(r^2 - 2h^2)} + \frac{1}{2(r^2 - 4h^2)} - \frac{1}{2r^2} \\ &= \frac{2}{r^2} \left( 1 - \frac{h^2}{r^2} \right)^{-1} - \frac{2}{r^2} \left( 1 - \frac{2h^2}{r^2} \right)^{-1} + \frac{1}{2r^2} \left( 1 - \frac{4h^2}{r^2} \right)^{-1} - \frac{1}{2r^2}. \end{aligned}$$

So,

$$\begin{aligned} (37) \quad A &= \frac{2}{r^2} \left( \frac{h^4}{r^4} + \frac{h^6}{r^6} + \frac{h^8}{r^8} + \frac{h^{10}}{r^{10}} + \frac{h^{12}}{r^{12}} + \dots \right) - \frac{2}{r^2} \left( \frac{8h^6}{r^6} + 16\frac{h^8}{r^8} + 32\frac{h^{10}}{r^{10}} + 64\frac{h^{12}}{r^{12}} + \dots \right) \\ &\quad + \frac{1}{2r^2} \left( 64\frac{h^6}{r^6} + 256\frac{h^8}{r^8} + 1024\frac{h^{10}}{r^{10}} + 4096\frac{h^{12}}{r^{12}} + \dots \right). \end{aligned}$$

$$B = h^4 \hat{\beta}_{1,n} \sum_{i=0}^2 \alpha_i \frac{i^2}{(r^2 - ih^2)^3} = h^4 \hat{\beta}_{1,n} \left( \frac{2}{(r^2 - h^2)^3} - \frac{4}{(r^2 - 2h^2)^3} \right)$$

$$= 2h^4 \frac{(r^2 - h^2)^2 \cdot (r^6 - 6r^2h^4 + 6h^6)}{(r^2 - 2h^2)^3 \cdot (r^4 + 4r^2h^2 + h^4)} \left( \frac{-r^6 + 6r^2h^4 - 6h^6}{(r^2 - h^2)^3 (r^2 - 2h^2)^3} \right)$$

$$= -2h^4 (r^6 - 6r^2h^4 + 6h^6)^2 \frac{1}{r^{12}} \left(1 - \frac{2h^2}{r^2}\right)^{-6} \frac{1}{r^4} \left[1 + \frac{h^2}{r^2} \left(4 + \frac{h^2}{r^2}\right)\right]^{-1} \frac{1}{r^2} \left[1 - \frac{h^2}{r^2}\right]^{-1}$$

$$= -\frac{2}{r^2} \frac{h^4}{r^4} \left(1 - \frac{6h^4}{r^4} + \frac{6h^6}{r^6}\right)^2 \left[1 - \frac{h^2}{r^2}\right]^{-1} \left(1 - \frac{2h^2}{r^2}\right)^{-6} \left[1 + \frac{h^2}{r^2} \left(4 + \frac{h^2}{r^2}\right)\right]^{-1}$$

$$= -\frac{2}{r^2} \frac{h^4}{r^4} \left(1 - 12\frac{h^4}{r^4} + 12\frac{h^6}{r^6} + 36\frac{h^8}{r^8} + \dots\right) \cdot \left(1 + \frac{h^2}{r^2} + \frac{h^4}{r^4} + \frac{h^6}{r^6} + \frac{h^8}{r^8} + \dots\right).$$

$$\left(1 + 12\frac{h^2}{r^2} + 84\frac{h^4}{r^4} + 448\frac{h^6}{r^6} + 2016\frac{h^8}{r^8} + \dots\right) \cdot \left(1 - 4\frac{h^2}{r^2} - \frac{h^4}{r^4} + 16\frac{h^4}{r^4} + 8\frac{h^6}{r^6} - 64\frac{h^6}{r^6} + \frac{h^8}{r^8} - 48\frac{h^8}{r^8} + \dots\right)$$

$$= -\frac{2}{r^2} \frac{h^4}{r^4} \left(1 - 12\frac{h^4}{r^4} + 12\frac{h^6}{r^6} + 36\frac{h^8}{r^8} + \dots\right) \cdot \left(1 + \frac{h^2}{r^2} + \frac{h^4}{r^4} + \frac{h^6}{r^6} + \frac{h^8}{r^8} + \dots\right).$$

$$\left(1 + 8\frac{h^2}{r^2} + 51\frac{h^4}{r^4} + 236\frac{h^6}{r^6} + 1021\frac{h^8}{r^8} + \dots\right)$$

$$= -\frac{2}{r^2} \frac{h^4}{r^4} \left(1 - 12\frac{h^4}{r^4} + 12\frac{h^6}{r^6} + 36\frac{h^8}{r^8} + \dots\right) \cdot \left(1 + 9\frac{h^2}{r^2} + 60\frac{h^4}{r^4} + 296\frac{h^6}{r^6} + 1317\frac{h^8}{r^8} + \dots\right).$$

So,

$$(38) \quad B = -\frac{2}{r^2} \frac{h^4}{r^4} \left(1 + 9\frac{h^2}{r^2} + 48\frac{h^4}{r^4} + 200\frac{h^6}{r^6} + 705\frac{h^8}{r^8} + \dots\right).$$

Using (37) and (38), we get

$$(39) \quad \left\| \hat{T}_n \right\|^2 \Big|_{x=h} = \frac{r}{\pi} (A + B) = \frac{r}{\pi} \left( \frac{2h^8}{r^{10}} + 50\frac{h^{10}}{r^{12}} + 512\frac{h^{12}}{r^{14}} + \dots \right).$$

From (35), we get

$$\begin{aligned} \left\| T_n \right\|^2 \Big|_{x=h} &= \sum_{i=0}^2 \alpha_i \left( \sum_{j=0}^2 \alpha_j \frac{r}{2\pi[r^2 - ijh^2]} + h^2 \frac{ri^2h^2}{\pi[r^2 - ih^2]^3} \right) \\ &\quad + h^2 \sum_{j=0}^2 \alpha_j \frac{rj^2h^2}{\pi[r^2 - jh^2]^3} + h^4 \frac{2r}{\pi(r^2 - h^2)^5} [r^2(r^2 + 4h^2) + h^4] \end{aligned}$$

$$\begin{aligned}
&= \frac{r}{\pi} \left[ \sum_{i=0}^2 \alpha_i \left\{ -\frac{1}{2r^2} + \frac{1}{(r^2 - ih^2)} - \frac{1}{2(r^2 - 2ih^2)} \right\} + 2h^4 \sum_{i=0}^2 \alpha_i \frac{i^2}{[r^2 - ih^2]^3} \right. \\
&\quad \left. + h^4 \frac{2[r^2(r^2 + 4h^2) + h^4]}{(r^2 - h^2)^5} \right] \\
&= \frac{r}{\pi} (A + C + D), \text{ say,}
\end{aligned}$$

where A is given by (37).

$$\begin{aligned}
C &= 2h^4 \sum_{i=0}^2 \alpha_i \frac{i^2}{[r^2 - ih^2]^3} = 2h^4 \left[ \frac{2}{[r^2 - h^2]^3} - \frac{4}{[r^2 - 2h^2]^3} \right] \\
&= - \frac{4h^4 (r^6 - 6r^2h^4 + 6h^6)}{(r^2 - h^2)^3 (r^2 - 2h^2)^3} \\
&= - \frac{4}{r^2} \frac{h^4}{r^4} \left( 1 - \frac{6h^4}{r^4} + \frac{6h^6}{r^6} \right) \left[ 1 - \frac{h^2}{r^2} \right]^{-3} \left( 1 - \frac{2h^2}{r^2} \right)^{-3} \\
&= - \frac{4}{r^2} \frac{h^4}{r^4} \left( 1 - \frac{6h^4}{r^4} + \frac{6h^6}{r^6} \right) \cdot \left( 1 + 9\frac{h^2}{r^2} + 48\frac{h^4}{r^4} + 198\frac{h^6}{r^6} + 684\frac{h^8}{r^8} + \dots \right),
\end{aligned}$$

so that

$$(40) \quad C = - \frac{4}{r^2} \frac{h^4}{r^4} \left( 1 + 9\frac{h^2}{r^2} + 42\frac{h^4}{r^4} + 150\frac{h^6}{r^6} + 450\frac{h^8}{r^8} + \dots \right).$$

$$D = h^4 \frac{2[r^2(r^2 + 4h^2) + h^4]}{(r^2 - h^2)^5} = \frac{2}{r^2} \frac{h^4}{r^4} \left( 1 + \frac{4h^2}{r^2} + \frac{h^4}{r^4} \right) \left[ 1 - \frac{h^2}{r^2} \right]^{-5},$$

so that

$$(41) \quad D = \frac{2}{r^2} \left( \frac{h^4}{r^4} + 9\frac{h^6}{r^6} + 36\frac{h^8}{r^8} + 100\frac{h^{10}}{r^{10}} + 225\frac{h^{12}}{r^{12}} + \dots \right).$$

Using (37), (40) and (41) we get

$$(42) \quad \|T_n\|^2 \Big|_{x=h} = \frac{r}{\pi} (A + C + D) = \frac{r}{\pi} \left( \frac{2h^8}{r^{10}} + 50\frac{h^{10}}{r^{12}} + 512\frac{h^{12}}{r^{14}} + \dots \right).$$

Comparing (39) and (42) we see,  $\|\hat{T}_n\|^2 \Big|_{x=h}$  and  $\|T_n\|^2 \Big|_{x=h}$  are equal

at least up to order of  $h^{12}$ .

(iii) At a general point  $x$ :

From (34),

$$\|\hat{T}_n\|^2 \Big|_x = \frac{r}{\pi} (\hat{A} + \hat{B}), \quad \text{where } \hat{A} = \hat{A}_1 + \hat{A}_2 + \hat{A}_3, \text{ with}$$

$$\begin{aligned} \hat{A}_1 &= \sum_{i=0}^2 \alpha_i \frac{-1}{2(r^2 - (x-h)(x-(1-i)h))} \\ &= \frac{1}{2((r^2 - x^2) + h(2x-h))} - \frac{1}{((r^2 - x^2) + hx)} + \frac{1}{2((r^2 - x^2) + h^2)}, \end{aligned}$$

$$\begin{aligned} \hat{A}_2 &= \sum_{i=0}^2 \alpha_i \frac{1}{(r^2 - x(x-(1-i)h))} \\ &= -\frac{1}{((r^2 - x^2) + hx)} + \frac{2}{(r^2 - x^2)} - \frac{1}{((r^2 - x^2) - hx)}, \end{aligned}$$

$$\begin{aligned} \hat{A}_3 &= \sum_{i=0}^2 \alpha_i \frac{-1}{2(r^2 - (x+h)(x-(1-i)h))} \\ &= \frac{1}{2((r^2 - x^2) + h^2)} - \frac{1}{((r^2 - x^2) - hx)} + \frac{1}{2((r^2 - x^2) - h(2x+h))}, \end{aligned}$$

so that

$$\begin{aligned} \hat{A} &= \hat{A}_1 + \hat{A}_2 + \hat{A}_3 = \\ &= \frac{2}{(r^2 - x^2)} + \frac{1}{2((r^2 - x^2) + h(2x-h))} + \frac{1}{2((r^2 - x^2) - h(2x+h))} + \\ &\quad + \frac{1}{((r^2 - x^2) + h^2)} - \frac{2}{((r^2 - x^2) + hx)} - \frac{2}{((r^2 - x^2) - hx)} \\ &= \frac{2}{(r^2 - x^2)} + \frac{1}{2(r^2 - x^2)} \left[ 1 + \frac{h(2x-h)}{(r^2 - x^2)} \right]^{-1} + \frac{1}{2(r^2 - x^2)} \left[ 1 - \frac{h(2x+h)}{(r^2 - x^2)} \right]^{-1} + \\ &\quad \frac{1}{(r^2 - x^2)} \left[ 1 + \frac{h^2}{r^2 - x^2} \right]^{-1} - \frac{2}{(r^2 - x^2)} \left[ 1 + \frac{hx}{r^2 - x^2} \right]^{-1} - \frac{2}{(r^2 - x^2)} \left[ 1 - \frac{hx}{r^2 - x^2} \right]^{-1}. \end{aligned}$$



since the first term gets canceled and the last two terms get combined, we get

$$\begin{aligned}\hat{A} = & \frac{1}{2(r^2-x^2)} \left[ \frac{h(2x-h)}{(r^2-x^2)} + \frac{h^2(2x-h)^2}{(r^2-x^2)^2} - \frac{h^3(2x-h)^3}{(r^2-x^2)^3} + \frac{h^4(2x-h)^4}{(r^2-x^2)^4} - \dots \right] \\ & + \frac{1}{2(r^2-x^2)} \left[ \frac{h(2x+h)}{(r^2-x^2)} + \frac{h^2(2x+h)^2}{(r^2-x^2)^2} + \frac{h^3(2x+h)^3}{(r^2-x^2)^3} + \frac{h^4(2x+h)^4}{(r^2-x^2)^4} + \dots \right] \\ & + \frac{1}{(r^2-x^2)} \left[ -\frac{h^2}{r^2-x^2} + \frac{h^4}{(r^2-x^2)^2} - \frac{h^6}{(r^2-x^2)^3} + \frac{h^8}{(r^2-x^2)^4} - \frac{h^{10}}{(r^2-x^2)^5} + \dots \right] \\ & - \frac{2}{(r^2-x^2)} \left[ \frac{2h^2x^2}{(r^2-x^2)^2} + \frac{2h^4x^4}{(r^2-x^2)^4} + \frac{2h^6x^6}{(r^2-x^2)^6} + \frac{2h^8x^8}{(r^2-x^2)^8} + \frac{2h^{10}x^{10}}{(r^2-x^2)^{10}} + \dots \right].\end{aligned}$$

Using  $(2x-h)^2 + (2x+h)^2 = 2(4x^2 + h^2)$ ,

$$- (2x-h)^3 + (2x+h)^3 = 2(12x^2h + h^3),$$

$$(2x-h)^4 + (2x+h)^4 = 2(16x^4 + 24x^2h^2 + h^4),$$

$$- (2x-h)^5 + (2x+h)^5 = 2(80x^4h + 40x^2h^3 + h^5),$$

$$(2x-h)^6 + (2x+h)^6 = 2(64x^6 + 240x^4h^2 + 60x^2h^4 + h^6),$$

$$- (2x-h)^7 + (2x+h)^7 = 2(448x^6h + 560x^4h^3 + \dots),$$

and  $(2x-h)^8 + (2x+h)^8 = 2(256x^8 + 1792x^6h^2 + \dots),$

we get

$$\begin{aligned}(43) \quad \hat{A} = & h^4 \left\{ \frac{2}{(r^2-x^2)^3} + \frac{12x^2}{(r^2-x^2)^4} + \frac{12x^4}{(r^2-x^2)^5} \right\} \\ & + h^6 \left\{ \frac{24x^2}{(r^2-x^2)^5} + \frac{80x^4}{(r^2-x^2)^6} + \frac{60x^6}{(r^2-x^2)^7} \right\} \\ & + h^8 \left\{ \frac{2}{(r^2-x^2)^5} + \frac{40x^2}{(r^2-x^2)^6} + \frac{240x^4}{(r^2-x^2)^7} + \frac{448x^6}{(r^2-x^2)^8} + \frac{252x^8}{(r^2-x^2)^9} \right\} + O(h^{10}).\end{aligned}$$

$$\hat{B} = h^2 \hat{\beta}_{1,n} \sum_{i=0}^2 \alpha_i \frac{(x-(1-i)h)^2}{(r^2-x(x-(1-i)h))^3}$$

$$\begin{aligned}
&= h^2 \hat{\beta}_{1,n} \left\{ -\frac{(x-h)^2}{((r^2-x^2)+hx)^3} + \frac{2x^2}{(r^2-x^2)^3} - \frac{(x+h)^2}{((r^2-x^2)-hx)^3} \right\} \\
&= h^2 \hat{\beta}_{1,n} \left\{ \frac{2x^2}{(r^2-x^2)^3} - \frac{(x-h)^2}{(r^2-x^2)^3} \left(1 + \frac{hx}{r^2-x^2}\right)^{-3} - \frac{(x+h)^2}{(r^2-x^2)^3} \left(1 - \frac{hx}{r^2-x^2}\right)^{-3} \right\}.
\end{aligned}$$

Using  $(x-h)^2 + (x+h)^2 = 2(x^2+h^2)$  and substituting the value of  $\hat{\beta}_{1,n}$ , from (33), we get

$$\begin{aligned}
\hat{B} &= -h^4 \left[ 1 + \frac{(r^2-x^2)^2}{(r^4+4r^2x^2+x^4)} \left\{ \frac{6x^2h^2}{(r^2-x^2)^2} + \frac{20x^4h^2}{(r^2-x^2)^3} + \frac{15x^6h^2}{(r^2-x^2)^4} + \frac{15x^4h^4}{(r^2-x^2)^4} \right. \right. \\
&\quad + \frac{42x^6h^4}{(r^2-x^2)^5} + \frac{28x^8h^4}{(r^2-x^2)^6} + \frac{28x^6h^6}{(r^2-x^2)^6} + \frac{72x^8h^6}{(r^2-x^2)^7} + \frac{45x^{10}h^6}{(r^2-x^2)^8} + \frac{45x^8h^8}{(r^2-x^2)^8} \\
&\quad \left. \left. + \frac{110x^{10}h^8}{(r^2-x^2)^9} + \frac{66x^{12}h^8}{(r^2-x^2)^{10}} + \dots \right\} \right] \\
&\quad \cdot \left[ \frac{2}{(r^2-x^2)^3} + \frac{12x^2}{(r^2-x^2)^4} + \frac{12x^4}{(r^2-x^2)^5} + \frac{12x^2h^2}{(r^2-x^2)^5} + \frac{40x^4h^2}{(r^2-x^2)^6} + \frac{30x^6h^2}{(r^2-x^2)^7} \right. \\
&\quad \left. + \frac{30x^4h^4}{(r^2-x^2)^7} + \frac{84x^6h^4}{(r^2-x^2)^8} + \frac{56x^8h^4}{(r^2-x^2)^9} + \frac{56x^6h^6}{(r^2-x^2)^9} + \frac{144x^8h^6}{(r^2-x^2)^{10}} + \frac{90x^{10}h^6}{(r^2-x^2)^{11}} + \dots \right].
\end{aligned}$$

After simplifying, we get

$$\begin{aligned}
(44) \quad \hat{B} &= -h^4 \left[ \frac{2}{(r^2-x^2)^3} + \frac{12x^2}{(r^2-x^2)^4} + \frac{12x^4}{(r^2-x^2)^5} \right] \\
&\quad -h^6 \left[ \frac{24x^2}{(r^2-x^2)^5} + \frac{80x^4}{(r^2-x^2)^6} + \frac{60x^6}{(r^2-x^2)^7} \right] -h^8 \left[ 2 \left\{ \frac{30x^4}{(r^2-x^2)^7} + \frac{84x^6}{(r^2-x^2)^8} + \frac{56x^8}{(r^2-x^2)^9} \right\} \right. \\
&\quad \left. + \frac{2}{(r^2-x^2)(r^4+4r^2x^2+x^4)} \left\{ \frac{6x^2}{(r^2-x^2)^2} + \frac{20x^4}{(r^2-x^2)^3} + \frac{15x^6}{(r^2-x^2)^4} \right\}^2 \right] + O(h^{10}).
\end{aligned}$$

By (43) and (44), we get  $\|\hat{T}_n\|^2 = \frac{r}{\pi} (\hat{A} + \hat{B})$ , so that

$$(45) \quad \|\hat{T}_n\|^2 = \frac{r}{\pi} h^8 \left[ \frac{2}{(r^2-x^2)^5} + \frac{40x^2}{(r^2-x^2)^6} + \frac{180x^4}{(r^2-x^2)^7} + \frac{280x^6}{(r^2-x^2)^8} + \frac{140x^8}{(r^2-x^2)^9} \right. \\ \left. - \frac{2}{(r^2-x^2)(r^4+4r^2x^2+x^4)} \left( \frac{6x^2}{(r^2-x^2)^2} + \frac{20x^4}{(r^2-x^2)^3} + \frac{15x^6}{(r^2-x^2)^4} \right)^2 \right] + O(h^{10}).$$

Now by (35),  $\|T_n\|^2 = \frac{r}{\pi} (\hat{A}+E+F)$ , where  $\hat{A}$  is given by (43). As

$$E = 2h^2 \sum_{i=0}^2 \alpha_i \frac{\{x-(1-i)h\}^2}{[r^2-\{x-(1-i)h\}x]^3}, \\ = 2h^2 \left[ \frac{2x^2}{(r^2-x^2)^3} - \frac{(x-h)^2}{(r^2-x^2)^3} \left( 1 + \frac{hx}{(r^2-x^2)} \right)^{-3} - \frac{(x+h)^2}{(r^2-x^2)^3} \left( 1 - \frac{hx}{(r^2-x^2)} \right)^{-3} \right],$$

we have

$$(46) \quad E = -2h^4 \left[ \left( \frac{2}{(r^2-x^2)^3} + \frac{12x^2}{(r^2-x^2)^4} + \frac{12x^4}{(r^2-x^2)^5} \right) \right. \\ \left. + \left( \frac{12x^2h^2}{(r^2-x^2)^5} + \frac{40x^4h^2}{(r^2-x^2)^6} + \frac{30x^6h^2}{(r^2-x^2)^7} \right) + \left( \frac{30x^4h^4}{(r^2-x^2)^7} + \frac{84x^6h^4}{(r^2-x^2)^8} + \frac{56x^8h^4}{(r^2-x^2)^9} \right) \right. \\ \left. + \left( \frac{56x^6h^6}{(r^2-x^2)^9} + \frac{144x^8h^6}{(r^2-x^2)^{10}} + \frac{90x^{10}h^6}{(r^2-x^2)^{11}} \right) + \dots \right].$$

Also

$$(47) \quad F = h^4 \frac{2[r^2(r^2+4x^2)+x^4]}{(r^2-x^2)^5} = h^4 \left[ \frac{2}{(r^2-x^2)^3} + \frac{12x^2}{(r^2-x^2)^4} + \frac{12x^4}{(r^2-x^2)^5} \right].$$

Hence, by (43), (46) and (47), we get

$$(48) \quad \|T_n\|^2 = \frac{r}{\pi} (\hat{A}+E+F) \\ = \frac{r}{\pi} h^8 \left[ \frac{2}{(r^2-x^2)^5} + \frac{40x^2}{(r^2-x^2)^6} + \frac{180x^4}{(r^2-x^2)^7} + \frac{280x^6}{(r^2-x^2)^8} + \frac{140x^8}{(r^2-x^2)^9} \right] + O(h^{10}).$$

Comparing (45) and (48), we get

$$\|T_n\|^2 - \|\hat{T}_n\|^2 = \frac{r}{\pi} h^8 \frac{2}{(r^2-x^2)(r^4+4r^2x^2+x^4)} \left( \frac{6x^2}{(r^2-x^2)^2} + \frac{20x^4}{(r^2-x^2)^3} + \frac{15x^6}{(r^2-x^2)^4} \right)^2 + \\ + O(h^{10}).$$

So, we conclude that  $\|T_n\|$ ,  $\|\hat{T}_n\|$ ,  $\|T_n\| - \|\hat{T}_n\|$  are of  $O(h^4)$ , for all  $x \neq 0$ , in  $(-r, r)$ .

Next let us consider Cowell's method with function evaluation at three points as the usual method, for solving numerically a differential equation (I), with some initial or boundary conditions, which is given by

$$(49) \quad \sum_{i=0}^2 \alpha_i y_{n-1+i} + h^2 \sum_{i=0}^2 \beta_i f_{n-1+i} = 0,$$

where  $\alpha_0 = -1$ ,  $\alpha_1 = 2$ ,  $\alpha_2 = -1$ ;  $\alpha_0 = 1/12$ ,  $\alpha_1 = 10/12$ ,  $\alpha_2 = 1/12$ .

Let

$$(50) \quad \sum_{i=0}^2 \alpha_i y_{n-1+i} + h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} f_{n-1+i} = 0$$

be the corresponding  $\beta$ -optimal method, where  $\hat{\beta}_{i,n}$  are the optimal coefficients at a point  $x_n$ ,  $1 \leq n \leq N-1$ .

Now the local truncation error functional  $T_n$  at a point  $x_n$  of the usual method (49) applied to the true solution  $y(x)$  is given by

$$T_n y = \sum_{i=0}^2 \alpha_i y(x_{n-1+i}) + h^2 \sum_{i=0}^2 \beta_i y''(x_{n-1+i}).$$

The representer for the local truncation error functional  $T_n$  is

$$\begin{aligned} & \sum_{i=0}^2 \bar{\alpha}_i K(t, \bar{x}_{n-1+i}) + h^2 \sum_{i=0}^2 \bar{\beta}_i D_2(t, \bar{x}_{n-1+i}), \text{ from which} \\ \|T_n\|^2 = & \left( \sum_{i=0}^2 \bar{\alpha}_i K(t, \bar{x}_{n-1+i}) + h^2 \sum_{i=0}^2 \bar{\beta}_i D_2(t, \bar{x}_{n-1+i}), \right. \\ & \left. \sum_{j=0}^2 \bar{\alpha}_j K(t, \bar{x}_{n-1+j}) + h^2 \sum_{j=0}^2 \bar{\beta}_j D_2(t, \bar{x}_{n-1+j}) \right), \end{aligned}$$

is given by

$$(51) \quad \|T_n\|^2 = \sum_{i=0}^2 \bar{\alpha}_i \left( \sum_{j=0}^2 \alpha_j K(x_{n-1+j}, \bar{x}_{n-1+i}) + h^2 \sum_{j=0}^2 \beta_j \overline{D2(x_{n-1+i}, \bar{x}_{n-1+j})} \right) \\ + h^2 \sum_{i=0}^2 \bar{\beta}_i \left( \sum_{j=0}^2 \alpha_j D2(x_{n-1+j}, \bar{x}_{n-1+i}) + h^2 \sum_{j=0}^2 \beta_j D2''(x_{n-1+j}, \bar{x}_{n-1+i}) \right).$$

Using the normal equations for obtaining  $\hat{\beta}_{j,n}$ ,  $j = 0(1)2$ ,  $\|\hat{T}_n\|^2$ , the square norm of local truncation error functional of the  $\beta$ -optimal method is given by

$$(52) \quad \|\hat{T}_n\|^2 = \sum_{i=0}^2 \bar{\alpha}_i \left( \sum_{j=0}^2 \alpha_j K(x_{n-1+j}, \bar{x}_{n-1+i}) + h^2 \sum_{j=0}^2 \hat{\beta}_{j,n} \overline{D2(x_{n-1+i}, \bar{x}_{n-1+j})} \right).$$

The normal equations for obtaining  $\hat{\beta}_{i,n}$ ,  $i = 0(1)2$  are given by

$$\sum_{i=0}^2 \alpha_i D2(x_{n-1+i}, \bar{x}_{n-1+k}) + h^2 \sum_{i=0}^2 \hat{\beta}_{i,n} D2''(x_{n-1+i}, \bar{x}_{n-1+k}) = 0, \quad k=0(1)2.$$

At  $x = 0$ , it gives the system of equations

$$D\beta = E,$$

where  $D = -h^2 \begin{bmatrix} D2''(-h, -h) & D2''(0, -h) & D2''(h, -h) \\ D2''(-h, 0) & D2''(0, 0) & D2''(h, 0) \\ D2''(-h, h) & D2''(0, h) & D2''(h, h) \end{bmatrix},$

$$\beta = \begin{bmatrix} \hat{\beta}_{1,n} & \hat{\beta}_{2,n} & \hat{\beta}_{3,n} \end{bmatrix}^T,$$

$$E = \begin{bmatrix} -D2(-h, -h) + 2D2(0, -h) - D2(h, -h) \\ -D2(-h, 0) + 2D2(0, 0) - D2(h, 0) \\ -D2(-h, h) + 2D2(0, h) - D2(h, h) \end{bmatrix}.$$

In  $H^2(C_r)$ -space, writing the values of  $D2$  and  $D2''$  at the nodal points, from (5) and (6), we get

$$D = \begin{bmatrix} a_1 & b_1 & c_1 \\ b_1 & b_1 & b_1 \\ c_1 & b_1 & a_1 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} e_1 \\ b_1 \\ e_1 \end{bmatrix},$$

$$\text{with } a_1 = \frac{2r(r^4 + 4r^2h^2 + h^4)}{(r^2 - h^2)^5}, \quad b_1 = \frac{2}{r^5}, \quad c_1 = \frac{2r(r^4 - 4r^2h^2 + h^4)}{(r^2 + h^2)^5}$$

$$\text{and } e_1 = \frac{r}{(r^2 - h^2)^3} + \frac{r}{(r^2 + h^2)^3}.$$

solving the above system of equations we get the solution

$$\hat{\beta}_{1,n} = \frac{b_1 - e_1}{2b_1 - c_1 - a_1}, \quad \hat{\beta}_{2,n} = \frac{2e_1 - c_1 - a_1}{2b_1 - c_1 - a_1}, \quad \hat{\beta}_{3,n} = \frac{b_1 - e_1}{2b_1 - c_1 - a_1}.$$

$$\text{Now, } b_1 - e_1 = \frac{2(r^4 - h^4)^3 - r^6(r^2 + h^2)^3 - r^6(r^2 - h^2)^3}{r^5(r^4 - h^4)^3},$$

$$2b_1 - c_1 - a_1 = \{4(r^4 - h^4)^5 - 2r^6(r^2 + h^2)^5(r^4 + 4r^2h^2 + h^4) - 2r^6(r^2 - h^2)^5(r^4 - 4r^2h^2 + h^4)\} / r^5(r^4 - h^4)^5,$$

$$2e_1 - c_1 - a_1 = \{[2r(r^2 - h^2)^2(r^2 + h^2)^5 + 2r(r^2 + h^2)^2(r^2 - h^2)^5 - 2r(r^2 + h^2)^5(r^4 + 4r^2h^2 + h^4) - 2r(r^2 - h^2)^5(r^4 - 4r^2h^2 + h^4)]\} / (r^4 - h^4)^5$$

$$\hat{\beta}_{1,n} = \hat{\beta}_{3,n} = \text{Num1/Din1} =$$

$$= \frac{2(r^4 - h^4)^5 - r^6(r^2 + h^2)^3(r^4 - h^4)^2 - r^6(r^2 - h^2)^3(r^4 - h^4)^2}{4(r^4 - h^4)^5 - 2r^6(r^2 + h^2)^5(r^4 + 4r^2h^2 + h^4) - 2r^6(r^2 - h^2)^5(r^4 - 4r^2h^2 + h^4)}.$$

$$\begin{aligned} \text{Din1} &= \left\{ 2(r^4 - h^4)^5 - 2r^6(r^2 + h^2)^5(r^4 + 4r^2h^2 + h^4) \right\} \\ &\quad + \left\{ 2(r^4 - h^4)^5 - 2r^6(r^2 - h^2)^5(r^4 - 4r^2h^2 + h^4) \right\} \\ &= 2(r^2 + h^2)^5 \left\{ -9r^8h^2 + 9r^6h^4 - 10r^4h^6 + 5r^2h^8 - h^{10} \right\} \\ &\quad + 2(r^2 - h^2)^5 \left\{ 9r^8h^2 + 9r^6h^4 + 10r^4h^6 + 5r^2h^8 + h^{10} \right\}. \end{aligned}$$

$$\text{Using } (r^2 + h^2)^5 + (r^2 - h^2)^5 = 2(r^{10} + 10r^6h^4 + 5r^2h^8)$$

$$\text{and } -(r^2 + h^2)^5 + (r^2 - h^2)^5 = -2(5r^8h^2 + 10r^4h^6 + h^{10}), \text{ we get}$$

$$\begin{aligned} \text{Din1} &= -36r^8h^2(5r^8h^2 + 10r^4h^6 + h^{10}) + 36r^6h^4(r^{10} + 10r^6h^4 + 5r^2h^8) \\ &\quad - 40r^4h^6(5r^8h^2 + 10r^4h^6 + h^{10}) + 20r^2h^8(r^{10} + 10r^6h^4 + 5r^2h^8) \end{aligned}$$

$$- 4h^{10}(5r^8h^2 + 10r^4h^6 + h^{10})$$

$$= - 144r^{16}h^4 - 180r^{12}h^8 - 76r^8h^{12} + 20r^4h^{16} - 4h^{20}, \text{ and}$$

$$\text{Num1} = 2(r^4 - h^4)^5 - r^6(r^4 - h^4)^2 \left\{ (r^2 + h^2)^3 + (r^2 - h^2)^3 \right\}$$

$$= 2(r^{20} - 5r^{16}h^4 + 10r^{12}h^8 - 10r^8h^{12} + 5r^4h^{16} - h^{20})$$

$$- r^6(r^8 - 2r^4h^4 + h^8)(2r^6 + 6r^2h^4)$$

$$= - 12r^{16}h^4 + 30r^{12}h^8 - 26r^8h^{12} + 10r^4h^{16} - 2h^{20}. \text{ Hence}$$

$$\begin{aligned} \hat{\beta}_{1,n} = \hat{\beta}_{3,n} &= \frac{- 12r^{16}h^4 + 30r^{12}h^8 - 26r^8h^{12} + 10r^4h^{16} - 2h^{20}}{- 144r^{16}h^4 - 180r^{12}h^8 - 76r^8h^{12} + 20r^4h^{16} - 4h^{20}} \\ &= \frac{1}{144} \left( 12 - 30 \frac{h^4}{r^4} + 26 \frac{h^8}{r^8} - 10 \frac{h^{12}}{r^{12}} + 2 \frac{h^{16}}{r^{16}} \right) \left( 1 + \frac{180}{144} \frac{h^4}{r^4} + \frac{76}{144} \frac{h^8}{r^8} - \frac{20}{144} \frac{h^{12}}{r^{12}} + \frac{4}{144} \frac{h^{16}}{r^{16}} \right)^{-1} \\ &= \left( \frac{1}{12} - \frac{5}{24} \frac{h^4}{r^4} + \frac{13}{72} \frac{h^8}{r^8} - \frac{5}{72} \frac{h^{12}}{r^{12}} + \frac{1}{72} \frac{h^{16}}{r^{16}} \right) \left[ 1 + \left\{ \frac{5}{4} \frac{h^4}{r^4} + \frac{19}{36} \frac{h^8}{r^8} - \frac{5}{36} \frac{h^{12}}{r^{12}} + \frac{1}{36} \frac{h^{16}}{r^{16}} \right\} \right]^{-1} \\ &= \left( \frac{1}{12} - \frac{5}{24} \frac{h^4}{r^4} + \frac{13}{72} \frac{h^8}{r^8} - \frac{5}{72} \frac{h^{12}}{r^{12}} + \frac{1}{72} \frac{h^{16}}{r^{16}} \right) \left[ 1 - \left\{ \frac{5}{4} \frac{h^4}{r^4} + \frac{19}{36} \frac{h^8}{r^8} - \frac{5}{36} \frac{h^{12}}{r^{12}} + \frac{1}{36} \frac{h^{16}}{r^{16}} \right\} \right. \\ &\quad \left. + \left\{ \frac{5}{4} \frac{h^4}{r^4} + \frac{19}{36} \frac{h^8}{r^8} - \frac{5}{36} \frac{h^{12}}{r^{12}} + \frac{1}{36} \frac{h^{16}}{r^{16}} \right\}^2 + \left\{ \frac{5}{4} \frac{h^4}{r^4} + \frac{19}{36} \frac{h^8}{r^8} - \frac{5}{36} \frac{h^{12}}{r^{12}} + \frac{1}{36} \frac{h^{16}}{r^{16}} \right\}^3 + \dots \right] \end{aligned}$$

$$(53) \quad \hat{\beta}_{1,n} = \hat{\beta}_{3,n} = \frac{1}{12} - \frac{5}{16} \frac{h^4}{r^4} + \frac{911}{1728} \frac{h^8}{r^8} - \frac{3815}{6912} \frac{h^{12}}{r^{12}} + \dots$$

$$\begin{aligned} \hat{\beta}_{2,n} = \frac{\text{Num2}}{\text{Den2}} &= \frac{(2r^6(r^2 - h^2)^2(r^2 + h^2)^5 + 2r^6(r^2 + h^2)^2(r^2 - h^2)^5 - \\ &\quad - 2r^6(r^2 + h^2)^5(r^4 + 4r^2h^2 + h^4) - 2r^6(r^2 - h^2)^5(r^4 - 4r^2h^2 + h^4))}{(4(r^4 - h^4)^5 - 2r^6(r^2 + h^2)^5(r^4 + 4r^2h^2 + h^4) - 2r^6(r^2 - h^2)^5(r^4 - 4r^2h^2 + h^4))} \end{aligned}$$

$$\begin{aligned} \text{Num2} &= 2r^6(r^2 + h^2)^5 \left\{ (r^2 - h^2)^2 - (r^4 + 4r^2h^2 + h^4) \right\} \\ &\quad + 2r^6(r^2 - h^2)^5 \left\{ (r^2 + h^2)^2 - (r^4 - 4r^2h^2 + h^4) \right\} \\ &= 12 \left( - 10r^{16}h^4 - 20r^{12}h^8 - 2r^8h^{12} \right). \end{aligned}$$

$$\text{Din2} = \text{Din1} = -144r^{16}h^4 - 180r^{12}h^8 - 76r^8h^{12} + 20r^4h^{16} - 4h^{20}.$$

$$\begin{aligned} \text{So, } \hat{\beta}_{2,n} &= \frac{12}{144} \left( 10 + 20 \frac{h^4}{r^4} + 2 \frac{h^8}{r^8} \right) \left[ 1 + \left\{ \frac{5}{4} \frac{h^4}{r^4} + \frac{19}{36} \frac{h^8}{r^8} - \frac{5}{36} \frac{h^{12}}{r^{12}} + \frac{1}{36} \frac{h^{16}}{r^{16}} \right\} \right]^{-1} \\ &= \frac{1}{12} \left( 10 + 20 \frac{h^4}{r^4} + 2 \frac{h^8}{r^8} \right) \times \\ &\times \left[ 1 - \left\{ \frac{5}{4} \frac{h^4}{r^4} + \frac{19}{36} \frac{h^8}{r^8} - \frac{5}{36} \frac{h^{12}}{r^{12}} + \frac{1}{36} \frac{h^{16}}{r^{16}} \right\} + \left\{ \frac{5}{4} \frac{h^4}{r^4} + \frac{19}{36} \frac{h^8}{r^8} - \frac{5}{36} \frac{h^{12}}{r^{12}} + \frac{1}{36} \frac{h^{16}}{r^{16}} \right\}^2 + \dots \right], \end{aligned}$$

so that

$$(54) \quad \hat{\beta}_{2,n} = \frac{1}{12} \left( 10 + \frac{15}{2} \frac{h^4}{r^4} - \frac{911}{72} \frac{h^8}{r^8} + \frac{7365}{288} \frac{h^{12}}{r^{12}} + \dots \right).$$

For the usual method (49), from (51) we get

$$\begin{aligned} \|T_n\|^2 \Big|_{x=0} &= \frac{r}{\pi} \left[ \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \alpha_j \frac{1}{2[r^2 - (j-1)(i-1)h^2]} + \right. \\ &\quad 2h^4 \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \beta_j \frac{(i-1)^2}{[r^2 - (j-1)(i-1)h^2]^3} + \\ &\quad \left. h^4 \sum_{i=0}^2 \beta_i \sum_{j=0}^2 \beta_j \frac{2[r^2\{r^2 + 4(i-1)(j-1)h^2\} + (i-1)^2(j-1)^2h^4]}{[r^2 - (j-1)(i-1)h^2]^5} \right] \\ &= \frac{r}{\pi} (X+Y+Z), \text{ say.} \end{aligned}$$

For the optimal method (50), from (52) we get

$$\begin{aligned} \|\hat{T}_n\|^2 \Big|_{x=0} &= \frac{r}{\pi} \left[ \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \alpha_j \frac{1}{2[r^2 - (j-1)(i-1)h^2]} + \right. \\ &\quad \left. h^4 \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \hat{\beta}_{j,n} \frac{(i-1)^2}{[r^2 - (j-1)(i-1)h^2]^3} \right] \\ &= \frac{r}{\pi} (X+W), \text{ say.} \end{aligned}$$

$$\text{Hence, } \|T_n\|^2 \Big|_{x=0} - \|\hat{T}_n\|^2 \Big|_{x=0} = \frac{r}{\pi} (Y+Z-W).$$



$$\begin{aligned}
 \text{Now, } X &= \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \alpha_j \frac{1}{2[r^2 - (j-1)(i-1)h^2]} \\
 &= \sum_{i=0}^2 \alpha_i \left[ \frac{-1}{2[r^2 + (i-1)h^2]} + \frac{1}{r^2} + \frac{-1}{2[r^2 - (i-1)h^2]} \right] \\
 &= \frac{1}{r^2 - h^2} - \frac{2}{r^2} + \frac{1}{r^2 + h^2} = \frac{1}{r^2} \left( 1 - \frac{h^2}{r^2} \right)^{-1} + \frac{1}{r^2} \left( 1 + \frac{h^2}{r^2} \right)^{-1} - \frac{2}{r^2},
 \end{aligned}$$

so that

$$(55) \quad X = 2\frac{h^4}{r^6} + 2\frac{h^8}{r^{10}} + 2\frac{h^{12}}{r^{14}} + 2\frac{h^{16}}{r^{18}} + \dots$$

$$\begin{aligned}
 Y &= 2h^4 \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \beta_j \frac{(i-1)^2}{[r^2 - (j-1)(i-1)h^2]^3} \\
 &= 2h^4 \sum_{i=0}^2 \alpha_i (i-1)^2 \left\{ \frac{\beta_0}{[r^2 + (i-1)h^2]^3} + \frac{\beta_1}{(r^2)^3} + \frac{\beta_2}{[r^2 - (i-1)h^2]^3} \right\} \\
 &= 2h^4 \left[ \alpha_1 \left\{ \frac{1}{12} \frac{1}{[r^2 - h^2]^3} + \frac{10}{12} \frac{1}{r^6} + \frac{1}{12} \frac{1}{[r^2 + h^2]^3} \right\} \right. \\
 &\quad \left. + \alpha_3 \left\{ \frac{1}{12} \frac{1}{[r^2 + h^2]^3} + \frac{10}{12} \frac{1}{r^6} + \frac{1}{12} \frac{1}{[r^2 - h^2]^3} \right\} \right] \\
 &= -2h^4 \left[ \frac{5}{3} \frac{1}{r^6} + \frac{1}{6} \frac{1}{r^6} \left( 1 - \frac{h^2}{r^2} \right)^{-3} + \frac{1}{6} \frac{1}{r^6} \left( 1 + \frac{h^2}{r^2} \right)^{-3} \right] \\
 &= -2h^4 \left[ \frac{5}{3} \frac{1}{r^6} + \frac{1}{6} \frac{1}{r^6} \left( 2 + 12\frac{h^4}{r^4} + 30\frac{h^6}{r^6} + 56\frac{h^8}{r^8} + \dots \right) \right], \text{ so that}
 \end{aligned}$$

$$(56) \quad Y = -4\frac{h^4}{r^6} - 4\frac{h^8}{r^{10}} - 10\frac{h^{12}}{r^{14}} - \frac{56}{3} \frac{h^{16}}{r^{18}} + \dots$$

$$\begin{aligned}
 Z &= h^4 \sum_{i=0}^2 \beta_i \sum_{j=0}^2 \beta_j \frac{2[r^2(r^2 + 4(i-1)(j-1)h^2) + (i-1)^2(j-1)^2h^4]}{[r^2 - (j-1)(i-1)h^2]^5} \\
 &= 2h^4 \sum_{i=0}^2 \beta_i \left\{ \beta_0 \frac{[r^2(r^2 - 4(i-1)h^2) + (i-1)^2h^4]}{[r^2 + (i-1)h^2]^5} + \beta_1 \frac{1}{r^6} + \right.
 \end{aligned}$$

$$\begin{aligned}
& + \beta_2 \frac{[r^2(r^2+4(i-1)h^2)+(i-1)^2h^4]}{[r^2-(i-1)h^2]^5} \Big\} \\
& = 2h^4 \left[ \beta_0 \left\{ \beta_0 \frac{r^2(r^2+4h^2)+h^4}{[r^2-h^2]^5} + \beta_1 \frac{1}{r^6} + \beta_2 \frac{r^2(r^2-4h^2)+h^4}{[r^2+h^2]^5} \right\} + \beta_1 \frac{1}{r^6} \sum_{i=0}^2 \beta_i + \right. \\
& \quad \left. \beta_2 \left\{ \beta_0 \frac{r^2(r^2-4h^2)+h^4}{[r^2+h^2]^5} + \beta_1 \frac{1}{r^6} + \beta_2 \frac{r^2(r^2+4h^2)+h^4}{[r^2-h^2]^5} \right\} \right]. \\
& \text{Using the fact that } \beta_0 = \beta_2, \text{ and } \sum_{i=0}^2 \beta_i = 1,
\end{aligned}$$

$$\begin{aligned}
Z &= 2h^4 \left[ 2\beta_0 \left\{ \beta_0 \frac{r^2(r^2+4h^2)+h^4}{[r^2-h^2]^5} + \beta_1 \frac{1}{r^6} + \beta_2 \frac{r^2(r^2-4h^2)+h^4}{[r^2+h^2]^5} \right\} + \beta_1 \frac{1}{r^6} \right] \\
&= 2 \frac{h^4}{r^6} \left[ \beta_1 + 2\beta_0 \left\{ \beta_0 \left( 1 + 4 \frac{h^2}{r^2} + \frac{h^4}{r^4} \right) \left( 1 - \frac{h^2}{r^2} \right)^{-5} + \beta_0 \left( 1 - 4 \frac{h^2}{r^2} + \frac{h^4}{r^4} \right) \left( 1 + \frac{h^2}{r^2} \right)^{-5} + \beta_1 \right\} \right].
\end{aligned}$$

$$\text{Using } \left( 1 - \frac{h^2}{r^2} \right)^{-5} + \left( 1 + \frac{h^2}{r^2} \right)^{-5} = 2 \left( 1 + 15 \frac{h^4}{r^4} + 70 \frac{h^8}{r^8} + 210 \frac{h^{12}}{r^{12}} + \dots \right)$$

$$\text{and } \left( 1 - \frac{h^2}{r^2} \right)^{-5} - \left( 1 + \frac{h^2}{r^2} \right)^{-5} = 2 \left( 5 \frac{h^2}{r^2} + 35 \frac{h^6}{r^6} + 126 \frac{h^{10}}{r^{10}} + \dots \right), \text{ we get}$$

$$(57) \quad Z = 2 \frac{h^4}{r^6} \left[ \beta_1 + 2\beta_0 \left\{ (\beta_1 + 2\beta_0) + \frac{h^4}{r^4} 72\beta_0 + 450\beta_0 \frac{h^8}{r^8} + 1568\beta_0 \frac{h^{12}}{r^{12}} + \dots \right\} \right].$$

Substituting the values of  $\beta_0$  and  $\beta_1$ , we get

$$Z = 2 \frac{h^4}{r^6} + 2 \frac{h^8}{r^{10}} + \frac{25}{2} \frac{h^{12}}{r^{14}} + \frac{98.4}{9} \frac{h^{16}}{r^{18}} + \dots$$

$$\begin{aligned}
W &= h^4 \sum_{i=0}^2 \alpha_i \sum_{j=0}^2 \hat{\beta}_{j,n} \frac{(i-1)^2}{[r^2-(j-1)(i-1)h^2]^3} \\
&= h^4 \sum_{i=0}^2 \alpha_i (i-1)^2 \left\{ \frac{\hat{\beta}_{0,n}}{[r^2+(i-1)h^2]^3} + \frac{\hat{\beta}_{1,n}}{(r^2)^3} + \frac{\hat{\beta}_{2,n}}{[r^2-(i-1)h^2]^3} \right\} \\
&= h^4 \left[ \hat{\beta}_{0,n} \left\{ \frac{\alpha_1}{[r^2-h^2]^3} + \frac{\alpha_3}{[r^2+h^2]^3} \right\} + \hat{\beta}_{1,n} \frac{\alpha_1 + \alpha_3}{r^6} + \hat{\beta}_{2,n} \left\{ \frac{\alpha_1}{[r^2+h^2]^3} + \frac{\alpha_3}{[r^2-h^2]^3} \right\} \right].
\end{aligned}$$

Since  $\hat{\beta}_{0,n} = \hat{\beta}_{2,n}$  and  $\alpha_1 = \alpha_3 = -1$ , we get

$$\begin{aligned}
 W &= -h^4 \left[ 2\hat{\beta}_{0,n} \left\{ \frac{1}{[r^2-h^2]^3} + \frac{1}{[r^2+h^2]^3} \right\} + 2\hat{\beta}_{1,n} \frac{1}{r^6} \right] \\
 &= -2\frac{h^4}{r^6} \left[ \hat{\beta}_{0,n} \left\{ \left(1-\frac{h^2}{r^2}\right)^{-3} + \left(1+\frac{h^2}{r^2}\right)^{-3} \right\} + \hat{\beta}_{1,n} \right] \\
 &= -2\frac{h^4}{r^6} \left[ 2\hat{\beta}_{0,n} \left( 1+6\frac{h^4}{r^4}+15\frac{h^8}{r^8}+28\frac{h^{12}}{r^{12}}+\dots \right) + \hat{\beta}_{1,n} \right] \\
 W &= -2\frac{h^4}{r^6} \left[ 2 \left\{ \left( \frac{1}{12} - \frac{5}{16} \frac{h^4}{r^4} + \frac{911}{1728} \frac{h^8}{r^8} - \frac{3815}{6912} \frac{h^{12}}{r^{12}} + \dots \right) \left( 1+6\frac{h^4}{r^4}+15\frac{h^8}{r^8}+28\frac{h^{12}}{r^{12}}+\dots \right) \right. \right. \\
 &\quad \left. \left. + \left( \frac{10}{12} + \frac{15}{24} \frac{h^4}{r^4} - \frac{911}{864} \frac{h^8}{r^8} + \frac{7365}{288} \frac{h^{12}}{r^{12}} + \dots \right) \right\} \right], \text{ [by (53) and (54)]} \\
 &= -2\frac{h^4}{r^6} \left[ \left( \frac{1}{6} + \frac{3}{8} \frac{h^4}{r^4} - \frac{169}{864} \frac{h^8}{r^8} + \frac{3512}{6912} \frac{h^{12}}{r^{12}} + \dots \right) \right. \\
 &\quad \left. + \left( \frac{5}{6} + \frac{5}{8} \frac{h^4}{r^4} - \frac{911}{864} \frac{h^8}{r^8} + \frac{7365}{288} \frac{h^{12}}{r^{12}} + \dots \right) \right], \text{ so that} \\
 (58) \quad W &= -2\frac{h^4}{r^6} - 2\frac{h^8}{r^{10}} + \frac{5}{2} \frac{h^{12}}{r^{14}} - \frac{180314}{6912} \frac{h^{16}}{r^{18}} + \dots
 \end{aligned}$$

By (56), (57), (58) we get

$$\|T_n\|^2 \Big|_{x=0} - \|\hat{T}_n\|^2 \Big|_{x=0} = \frac{r}{\pi} (Y+Z-W) = \frac{r}{\pi} \left[ \frac{180314}{6912} \frac{h^{16}}{r^{18}} + o(h^{16}) \right].$$

Thus we conclude that at  $x = 0$ ,  $\|T_n\| - \|\hat{T}_n\|$  is of  $O(h^8)$ .

Because of the complexity of calculation, we are omitting the similar evaluations for the methods (49) and (50) at  $x=h$ , and at a general point  $x$ . Similar results could also be obtained for other optimal methods discussed in Chapter 1.

## CHAPTER - 5

### OPTIMAL MULTISTEP METHODS IN $L^2(\hat{C}_r)$ SPACE

#### 5.1 Introduction

Similar to  $H^2(C_r)$  space, in  $L^2(\hat{C}_r)$  space the norm is based on area integral on the disk  $D_r$  and so the points inside  $C_r$  get more weightage than the points on the boundary of  $C_r$ . Consequently the optimal multistep methods obtained in  $L^2(\hat{C}_r)$  space are expected to result in a better distributed behavior in the central region of a subinterval  $[a,b]$  of  $(-r,r)$ . In set theoretic notation, we note that

$$L^2(\hat{C}_r) \supset H^2(C_r).$$

In this chapter we study the optimal multistep methods discussed in Chapter 1 for the Hilbert space  $L^2(\hat{C}_r)$ . For the purpose of numerical illustration, as usual methods, we have taken Störmer's method with function evaluation at one point and Cowell's method with function evaluation at three points for boundary value problems and Cowell's method with function evaluation at three points and Störmer's method with function evaluation at five points for initial value problems. The interpolatory functions tell that the optimal methods admit functions of higher growth in  $L^2(\hat{C}_r)$  space than in  $H^2(C_r)$  space.

In section 5.2, we prove that the second derivative of the kernel function of  $L^2(\hat{C}_r)$  space is the complex conjugate of the representer for the second derivative evaluation functional in  $L^2(\hat{C}_r)$  space. We devote section 4.3 to the discussion of optimal

multistep methods in  $L^2(\hat{C}_r)$  space and their numerical implementation, section 4.4 to the discussion of optimal multistep methods interpolatory for polynomials of certain degree and their numerical illustrations and section 4.5 to the discussion of optimal multistep methods interpolatory for linearly independent functions and their numerical illustrations. In section 4.6, we describe the limiting behavior of the optimal coefficients in the optimal methods in  $L^2(\hat{C}_r)$  space as  $r \rightarrow \infty$ .

## 5.2 The Hilbert space $L^2(\hat{C}_r)$

The space  $L^2(D_r)$  consists of measurable functions of complex variable  $z$  defined on the disc  $D_r = \{z : |z| < r\}$  which are square integrable in  $D_r$ . The space  $L^2(D_r)$  turns out to be a complex Hilbert space, with the inner product

$$(h, k) = \iint_{D_r} h(z) \overline{k(z)} \, dx dy, \quad h, k \in L^2(D_r).$$

The subspace of  $L^2(D_r)$ , consisting of functions of complex variable  $z$  ( $z = x + iy$ ) which are analytic within the disc  $D_r = \{z : |z| < r\}$ , is a closed subspace and has infinite dimension. This subspace turns out to be a complex Hilbert space  $L^2(\hat{C}_r)$  with the induced inner product given by

$$(1) \quad (f, g) = \iint_{D_r} f(z) \overline{g(z)} \, dx dy, \quad f, g \in L^2(C_r).$$

The space  $L^2(\hat{C}_r)$  possesses a reproducing kernel function given by

$$(2) \quad K(z, \bar{t}) = \frac{r^2}{\pi(r^2 - z\bar{t})^2},$$

and is separable with a complete orthonormal sequence  $\{\psi_k\}$  of functions given by

$$(3) \quad \psi_k(z) = \frac{z^k}{\sqrt{\pi} r^{k+1}}, \quad k = 0, 1, 2, \dots$$

As the space  $L^2(\hat{C}_r)$  possesses a reproducing kernel function, the linear functional for point evaluation:

$$L_z : f \longrightarrow f(z), \quad f \in L^2(\hat{C}_r), \quad z \in D_r$$

is a bounded linear functional (Aronszajn [4]).

Also the linear functional for  $k$ -th derivative evaluation

$$Dk_z : f \longrightarrow f^k(z), \quad f \in L^2(\hat{C}_r), \quad z \in D_r,$$

where  $k$  is a positive integer, is a bounded linear functional (Brij Bhusan [27]):

Theorem 1 : For  $z \in D_r$  and  $k$  any positive integer  $Dk_z$  is a bounded linear functional in  $L^2(\hat{C}_r)$  with

$$\| Dk_z \| \leq k! \sqrt{\frac{r}{2\pi}} \left( \frac{2}{r-|z|} \right)^{k+\frac{3}{2}}.$$

Now we shall show that the complex conjugate of the second derivative of the kernel is the representer for the second derivative evaluation functional in  $L^2(\hat{C}_r)$  space. We have

$$\overline{\frac{\partial^2}{\partial z^2} K(z, \bar{t})} \Big|_{z=z_0} = \frac{6r^2 t^2}{\pi(r^2 - \bar{z}_0 t)^4}.$$

Theorem 2: The function

$$D2(t, \bar{z}_0) = \frac{6r^2 t^2}{\pi(r^2 - \bar{z}_0 t)^4}$$

is the representer for the 2-nd derivative evaluation functional at  $z_0 \in D_r$ .

Proof: The only singularity of  $D_2(t, \bar{z}_0)$  is a pole of order 4 at  $t = \frac{r^2}{\bar{z}_0}$ . As  $z_0 \in D_r$ ,  $\frac{r^2}{\bar{z}_0}$  lies outside  $C_r = \{z : |z| = r\}$ . Hence the function  $D_2(t, \bar{z}_0)$  is analytic inside  $D_r$  and hence it belongs to  $L^2(\hat{C}_r)$ . We have to show that

$$f''(z_0) = \left( f(t) , \frac{6r^2 t^2}{\pi(r^2 - \bar{z}_0 t)^4} \right).$$

$$\begin{aligned} \text{Now, } \left( f(t) , \frac{6r^2 t^2}{\pi(r^2 - \bar{z}_0 t)^4} \right) &= \iint_{D_r} f(t) \frac{6r^2 \bar{t}^2}{\pi(r^2 - z_0 \bar{t})^4} dx dy \\ &= \int_{\rho=0}^r \int_{\theta=0}^{2\pi} f(\rho e^{i\theta}) \frac{6r^2 \rho^2 e^{-2i\theta}}{\pi(r^2 - z_0 \rho e^{-i\theta})^4} \rho d\theta d\rho, \quad (\text{putting } t = \rho e^{i\theta}) \\ &= \int_{\rho=0}^r \left\{ \int_{\theta=0}^{2\pi} f(\rho e^{i\theta}) \frac{6r^2}{\pi \left( \frac{r^2}{\rho^2} \rho e^{i\theta} - z_0 \right)^4} \frac{(\rho e^{i\theta})^2}{\rho^4} \frac{\rho i e^{i\theta} d\theta}{i e^{i\theta}} \right\} d\rho \\ &= \int_{\rho=0}^r \left\{ \int_{C_r} f(t) \frac{1}{\pi i} \frac{6r^2}{\left( t - z_0 \frac{\rho^2}{r^2} \right)^4} \frac{t^2}{\rho^3} \left( \frac{\rho^2}{r^2} \right)^4 \frac{1}{t} dt \right\} d\rho \\ &= \int_{\rho=0}^r \frac{6\rho^5}{r^6} \left\{ \int_{C_r} \frac{1}{\pi i} \frac{f(t)t}{\left( t - z_0 \frac{\rho^2}{r^2} \right)^4} dt \right\} d\rho \\ &= \int_{\rho=0}^r \frac{2\rho^5}{r^6} \left\{ \frac{3!}{2\pi i} \int_{C_r} \frac{f(t)t}{(t - t_0)^4} dt \Big|_{t_0 = z_0 \frac{\rho^2}{r^2}} \right\} d\rho. \end{aligned}$$

By using Cauchy integral formula for third derivative, we get

$$\left( f(t) , \frac{6r^2 t^2}{\pi(r^2 - \bar{z}_0 t)^4} \right) = \int_{\rho=0}^r \frac{2\rho^5}{r^6} [f(z)z]^{(3)} \Big|_{z=z_0 \frac{\rho^2}{r^2}} d\rho$$

$$= \int_{\rho=0}^r \frac{2\rho^5}{r^6} [zf^{(3)}(z) + 3f''(z)] \Big|_{z=z_0 \frac{\rho^2}{r^2}} d\rho.$$

If  $z_0 = 0$ , the last integral equals to

$$\int_{\rho=0}^r \frac{2\rho^5}{r^6} [0 \cdot f^{(3)}(0) + 3f''(0)] d\rho = 3f''(0) \int_{\rho=0}^r \frac{2\rho^5}{r^6} d\rho = f''(0).$$

On the other hand, if  $z_0 \neq 0$ , the last integral equals

$$\begin{aligned} & \int_{\rho=0}^r [f(z)z]^{(3)} \Big|_{z=z_0 \frac{\rho^2}{r^2}} \frac{\rho^4}{z_0 r^4} \frac{2\rho}{r^2} z_0 d\rho \\ &= \int_{z=0}^{z_0} [f(z)z]^{(3)} \frac{1}{z_0} \frac{z^2}{z_0^2} dz = \frac{1}{z_0^3} \int_{z=0}^{z_0} [f(z)z]^{(3)} z^2 dz \\ &= \frac{1}{z_0^3} \left[ [f(z)z]'' z^2 \Big|_{z=0}^{z_0} - \int_{z=0}^{z_0} [f(z)z]'' 2z dz \right] \quad (\text{Integrating by parts}) \\ &= \frac{1}{z_0^3} \left[ \left( [f(z)z]'' z^2 \right) \Big|_{z=0}^{z_0} - \left( [f(z)z]' 2z \right) \Big|_{z=0}^{z_0} + \left( 2[f(z)z] \right) \Big|_{z=0}^{z_0} \right] \\ &= \frac{1}{z_0^3} \left[ \left( f''(z_0) z_0 + 2f'(z_0) \right) z_0^2 - \left( f'(z_0) z_0 + f(z_0) \right) 2z_0 + 2f(z_0) z_0 \right] \\ &= f''(z_0). \quad \text{Hence the proof.} \end{aligned}$$

In  $L^2(\hat{C}_r)$  space, we have the following formulae:

$$(4) \quad K(z, \bar{t}) = \frac{r^2}{\pi(r^2 - z\bar{t})^2},$$

$$(5) \quad D_2(t, \bar{z}) = \overline{\frac{\partial^2}{\partial z^2} K(z, \bar{t})} = \frac{6r^2}{\pi} \frac{t^2}{(r^2 - t\bar{z})^4},$$



$$(6) \quad D_2''(t, \bar{z}) = \frac{\partial^2}{\partial t^2} D_2(t, \bar{z}) = \frac{12r^2}{\pi} \frac{1}{(r^2 - t\bar{z})^4} \left[ 1 + \frac{8t\bar{z}}{(r^2 - t\bar{z})} + \frac{10t^2\bar{z}^2}{(r^2 - t\bar{z})^2} \right].$$

As in  $H^2(C_r)$  space, in  $L^2(\hat{C}_r)$  space we shall implement some optimal multistep methods discussed in Chapter 1, especially (i)  $\beta$ -optimal methods (ii)  $\alpha$ -optimal methods, (iii)  $\beta$ -optimal methods with restriction (10) in the sequel, (iv)  $\beta$ -optimal methods interpolatory for polynomials of certain degree, and (v)  $\beta$ -optimal methods interpolatory for certain linearly independent functions corresponding to Cowell's usual method with function evaluation at three points, for the 24 BVP-s and 24 IVP-s with differential equations and solutions 1-24 listed in Chapter 4. We also implement  $\beta$ -optimal methods corresponding to Stormer's usual explicit method with function evaluation at five points on the same 24 IVP's and  $\beta$ -optimal methods corresponding to Stormer's usual method with function evaluation at one point for the same 24 BVP's. As the solutions of these 24 equations have singularities of different kinds at various locations, they require, for various ranges of intervals, different domains of space  $L^2(\hat{C}_r)$  for showing better performance of several optimal methods than the corresponding usual method. The solutions of the equations 1-9 and 19-24 are having singularities near the boundary of the domain of  $L^2(\hat{C}_r)$  space with  $r=2.01$ . The solutions of the equations 10-12 are entire functions and the solutions of the equations 13-18 are having singularities far from the boundary of the domain of  $L^2(\hat{C}_r)$  space with  $r = 2.01$ . The optimal methods are performing worse than the corresponding usual methods for the equations 10-18 with  $r=2.01$ . However with larger values of  $r$  and with larger intervals, optimal methods

perform better than the corresponding usual methods for the equations 10-18. The key symbols used in the tables of this chapter are the same as given in Chapter 4.

### 5.3 Optimal Multistep Methods in $L^2(\hat{C}_r)$ -Space

In  $L^2(\hat{C}_r)$  space, to determine the optimal coefficients  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k$ , of the  $\beta$ -optimal multistep method

$$(7) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} f_{n+k-j} = 0,$$

where  $\alpha_j$ 's are prefixed according to some consistent and stable known usual method with highest degree polynomial precision

$$(8) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0,$$

with reference to equation (11) of Chapter 1, we have the system of normal equations given by

$$\hat{C} \hat{b} = \hat{d},$$

where

$$\hat{b} = h^2 \left( \hat{\beta}_{\delta_{t_0},n}, \dots, \hat{\beta}_{k,n} \right)^T,$$

$$\hat{C}_{ij} = \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^4} \left[ 1 + \frac{8x_{n+k-j} \bar{x}_{n+k-1}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})} + \frac{10x_{n+k-j}^2 \bar{x}_{n+k-1}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2} \right],$$

$$i, j = \delta_{t_0}(1)k,$$

and

$$\hat{d}_i = -\frac{6r^2}{\pi} \sum_{j=0}^k \alpha_j \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^4}, \quad i = \delta_{t_0}(1)k.$$

This system of equations may be solved for  $h^2 \hat{\beta}_{j,n}$ ,  $j = \delta_{t_0}(1)k$ .

The following theorem characterizes the  $\beta$ -optimal multistep method (7) in  $L^2(\hat{C}_r)$  space.

Theorem 3: The optimal multistep method (7) in which  $\alpha_j$ 's are prefixed, and optimization is done with respect to  $\beta_j$ 's in  $L^2(\hat{C}_r)$  space is characterized by that it is locally interpolatory for functions

$$\left\{ y_i(x) = \frac{x^2}{(r^2 - x \bar{x}_{n+k-1})^4}, \quad i = \delta_{t_0}(1)k \right\}.$$

Proof: The proof follows from Theorem 1 of Chapter 1, using the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$  and  $D_2''(t, \bar{z})$  from (4), (5) and (6).

Corollary 1: If  $x_{n+k-1} = 0$ , for some  $i = \delta_{t_0}(1)k$ , then the optimal method (7) is consistent.

Proof: The proof is similar to that of Corollary 1 of Chapter 4.

In the following tables we are presenting some numerical results for the  $\beta$ -optimal methods corresponding to Stormer's 1-point, Cowell's 3-points and Stormer's 5-points methods.

Table - 1a

$x_n$	$h^2 \hat{\beta}_{1n}$	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.1091E-01	.6359E-03	.3283E-03
-1.6	.1050E-01	.3682E-04	.1918E-04
-1.5	.1031E-01	.4136E-05	.2222E-05
-1.4	.1021E-01	.7017E-06	.3929E-06
-1.3	.1015E-01	.1581E-06	.9278E-07
-1.2	.1011E-01	.4387E-07	.2709E-07
-1.1	.1008E-01	.1429E-07	.9312E-08
-1.0	.1006E-01	.5286E-08	.3647E-08
-.9	.1005E-01	.2173E-08	.1590E-08
-.8	.1004E-01	.9773E-09	.7587E-09
-.7	.1003E-01	.4755E-09	.3914E-09
-.6	.1002E-01	.2486E-09	.2164E-09
-.5	.1001E-01	.1393E-09	.1275E-09
-.4	.1001E-01	.8392E-10	.8000E-10
-.3	.1001E-01	.5477E-10	.5371E-10
-.2	.1000E-01	.3937E-10	.3918E-10
-.1	.1000E-01	.3183E-10	.3182E-10
.0	.1000E-01	.2957E-10	.2957E-10
.1	.1000E-01	.3183E-10	.3182E-10
.2	.1000E-01	.3937E-10	.3918E-10
.3	.1001E-01	.5477E-10	.5371E-10
.4	.1001E-01	.8392E-10	.8000E-10
.5	.1001E-01	.1393E-09	.1275E-09
.6	.1002E-01	.2486E-09	.2164E-09
.7	.1003E-01	.4755E-09	.3914E-09
.8	.1004E-01	.9773E-09	.7587E-09
.9	.1005E-01	.2173E-08	.1590E-08
1.0	.1006E-01	.5286E-08	.3647E-08
1.1	.1008E-01	.1429E-07	.9312E-08
1.2	.1011E-01	.4387E-07	.2709E-07
1.3	.1015E-01	.1581E-06	.9278E-07
1.4	.1021E-01	.7017E-06	.3929E-06
1.5	.1031E-01	.4136E-05	.2222E-05
1.6	.1050E-01	.3682E-04	.1918E-04
1.7	.1091E-01	.6359E-03	.3283E-03

Table for optimal  $\beta$ 's and the square norm of LTE functionals for Störmer's 1-point method and the corresponding  $\beta$ -optimal method in  $L^2(\text{Cr})$ -space, at the nodal points, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_1=-1$ ,  $\alpha_2=2$ ,  $\alpha_3=-1$ ;  $\beta_1=1$ .

Table 1a shows that the optimal  $\hat{\beta}_{1n}$ 's are depending on nodal points and that in a neighborhood of the origin, the optimal  $\hat{\beta}_{1n}$ 's are nearly equal to usual  $\beta_1$ . At a point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ , at  $x=0$ ,  $\hat{\beta}_{1n}$  and  $\beta_1$ , as well as  $\|\hat{T}_n\|^2$  and  $\|T_n\|^2$  are numerically equal.

Table - 1b

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	3	.275334E-02	2	.113882E-03
2	3	.137251E-02	3	.759100E-03
3	3	.425201E-02	3	.660814E-03
4	3	.226162E-02	3	.390301E-03
5	3	.547598E-02	3	.288221E-02
6	3	.110208E-02	3	.266344E-03
7	3	.269263E-03	3	.691810E-04
8	3	.152372E-02	2	.564563E-04
9	3	.829285E-03	2	.514084E-04
10	1	.122695E+00	1	.756851E-01
11	1	.163739E+00	1	.428818E-01
12	1	.159277E+00	1	.396641E-01
13	2	.262515E-03	2	.510016E-04
14	2	.190524E-03	2	.154842E-04
15	2	.279694E-03	2	.104465E-03
16	2	.207536E-03	2	.588077E-04
17	2	.237192E-05	2	.102788E-05
18	2	.200635E-04	2	.677542E-05
19	3	.486991E-02	3	.717588E-03
20	4	.601963E-02	3	.169067E-02
21	3	.473819E-02	3	.413338E-03
22	3	.268350E-02	3	.553183E-03
23	3	.474838E-02	3	.326285E-02
24	3	.924705E-03	3	.159715E-03

Table for number of iterations for convergence of the solution and the average discretisation errors using Störmer's 1-point usual and the corresponding  $\beta$ -optimal methods in  $L^2(\bar{C}_r)$ -space, for 24 BVP-s.

The results of Table 1b are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$  for BVP's with equations 1-9, 19-24; and with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$  for BVP's with equations 10-18. The numerical results show that this  $\beta$ -optimal method is two decimal places better on equation 8; one decimal place better on equations 1-4, 6, 7, 9-14, 16, 18, 19, 21 and 22, and just better on equations 5, 15, 17, 20, 23 and 24.

Table - 2a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.6057E-03	.9298E-02	-.1741E-03	.9730E-02	.2487E-04	.1748E-05
-1.6	.6882E-03	.8839E-02	.3942E-03	.9922E-02	.4108E-06	.3551E-07
-1.5	.7345E-03	.8640E-02	.5956E-03	.9970E-02	.1811E-07	.1882E-08
-1.4	.7630E-03	.8535E-02	.6884E-03	.9986E-02	.1452E-08	.1808E-09
-1.3	.7817E-03	.8473E-02	.7384E-03	.9993E-02	.1746E-09	.2615E-10
-1.2	.7948E-03	.8433E-02	.7683E-03	.9996E-02	.2820E-10	.5110E-11
-1.1	.8042E-03	.8406E-02	.7876E-03	.9998E-02	.5706E-11	.1260E-11
-1.0	.8113E-03	.8387E-02	.8007E-03	.9999E-02	.1380E-11	.3736E-12
-.9	.8168E-03	.8373E-02	.8100E-03	.9999E-02	.3865E-12	.1287E-12
-.8	.8210E-03	.8362E-02	.8168E-03	.1000E-01	.1223E-12	.5013E-13
-.7	.8244E-03	.8354E-02	.8218E-03	.1000E-01	.4305E-13	.2158E-13
-.6	.8270E-03	.8347E-02	.8256E-03	.1000E-01	.1664E-13	.1007E-13
-.5	.8291E-03	.8343E-02	.8283E-03	.1000E-01	.6986E-14	.5002E-14
-.4	.8307E-03	.8339E-02	.8303E-03	.1000E-01	.3262E-14	.2701E-14
-.3	.8319E-03	.8336E-02	.8317E-03	.1000E-01	.1700E-14	.1565E-14
-.2	.8327E-03	.8335E-02	.8326E-03	.1000E-01	.1011E-14	.9879E-15
-.1	.8331E-03	.8334E-02	.8331E-03	.1000E-01	.6583E-15	.6567E-15
.0	.8333E-03	.8333E-02	.8333E-03	.1000E-01	.5686E-15	.5686E-15
.1	.8331E-03	.8334E-02	.8331E-03	.1000E-01	.5920E-15	.5904E-15
.2	.8326E-03	.8335E-02	.8327E-03	.1000E-01	.9090E-15	.8863E-15
.3	.8317E-03	.8336E-02	.8319E-03	.1000E-01	.1656E-14	.1520E-14
.4	.8303E-03	.8339E-02	.8307E-03	.1000E-01	.3280E-14	.2718E-14
.5	.8283E-03	.8343E-02	.8291E-03	.1000E-01	.7039E-14	.5055E-14
.6	.8256E-03	.8347E-02	.8270E-03	.1000E-01	.1669E-13	.1013E-13
.7	.8218E-03	.8354E-02	.8244E-03	.1000E-01	.4311E-13	.2164E-13
.8	.8168E-03	.8362E-02	.8210E-03	.1000E-01	.1223E-12	.5009E-13
.9	.8100E-03	.8373E-02	.8168E-03	.9999E-02	.3864E-12	.1286E-12
1.0	.8007E-03	.8387E-02	.8113E-03	.9999E-02	.1380E-11	.3736E-12
1.1	.7876E-03	.8406E-02	.8042E-03	.9998E-02	.5706E-11	.1260E-11
1.2	.7683E-03	.8433E-02	.7948E-03	.9996E-02	.2820E-10	.5110E-11
1.3	.7384E-03	.8473E-02	.7817E-03	.9993E-02	.1746E-09	.2615E-10
1.4	.6884E-03	.8535E-02	.7630E-03	.9986E-02	.1452E-08	.1808E-09
1.5	.5956E-03	.8640E-02	.7345E-03	.9970E-02	.1811E-07	.1882E-08
1.6	.3942E-03	.8839E-02	.6882E-03	.9922E-02	.4108E-06	.3551E-07
1.7	-.1741E-03	.9298E-02	.6057E-03	.9730E-02	.2487E-04	.1748E-05

Optimal  $\beta$ 's, their row-sum and the square norm of LTE functionals for Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $L^2(\text{Cr})$ -space, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ;  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ .

Table 2a shows that the optimal  $\hat{\beta}_{1n}$ 's are depending on the nodal points and that at a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ . In a neighborhood of the origin, the optimal  $\hat{\beta}_{1n}$ 's are nearly equal to the corresponding usual  $\beta_1$ 's, and  $\|\hat{T}_n\|^2$  and  $\|T_n\|^2$  are very close.

Table - 2b(1)

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.283465E-04
2	2	.610703E-04	2	.535678E-04
3	3	.430195E-03	2	.221408E-05
4	2	.133292E-03	2	.395361E-04
5	2	.348263E-03	2	.228185E-03
6	3	.127772E-03	2	.692163E-05
7	2	.313175E-04	2	.211398E-05
8	2	.133523E-03	2	.121197E-04
9	2	.735570E-04	2	.265589E-05
10	1	.506252E-03	1	.209051E-02
11	1	.351768E-03	1	.134703E-01
12	1	.331492E-03	1	.132998E-01
13	2	.698086E-06	2	.363818E-04
14	2	.451879E-06	2	.328348E-04
15	2	.869978E-06	2	.262075E-04
16	2	.592198E-06	2	.235926E-04
17	1	.544969E-08	2	.101327E-06
18	1	.382493E-07	2	.105808E-05
19	3	.509406E-03	2	.234311E-04
20	3	.464907E-03	2	.197006E-04
21	2	.304134E-03	2	.737658E-04
22	2	.208304E-03	2	.553184E-04
23	2	.300455E-03	2	.226000E-03
24	2	.287372E-04	2	.394222E-04

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $L^2(\text{Cr})$ -space, for 24 BVP-s.

Numerical results of Table 2b(i) are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$  and  $h=.1$  revealing that as compared to usual method,  $\beta$ -optimal method performs better on BVP's with equations 1-9 and 19-23, but worse on BVP's with equations 10-18 and 24 .

The results of the following Table 2b(ii) are obtained with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$  for BVP's with equations 10-18, and with  $r=2.01$ ,  $a=-1.9$ ,  $b=1.9$ ,  $n=38$ ,  $h=.1$  for BVP with equation 24. Tables 2b(i) and 2b(ii) reveal that as compared to usual method,  $\beta$ -optimal method for BVP is one decimal place better on eqns. 1,3,4,6-10,13-22 and just better on eqns. 2,5,11,12,23,24.

Table - 2b(ii)

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
10	1	.170782E-02	1	.149522E-03
11	1	.738872E-03	1	.133189E-03
12	1	.718937E-03	1	.125355E-03
13	2	.298705E-05	1	.276705E-07
14	2	.167071E-05	1	.306032E-06
15	2	.392662E-05	2	.556663E-06
16	2	.236517E-05	1	.128682E-06
17	1	.257920E-07	1	.668273E-08
18	1	.151826E-06	1	.455183E-07
24	3	.210510E-03	3	.115611E-03

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $L^2(C_r)$ -space, for BVP-s 10-18 and 24.

Table - 2c

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.155059E-02
2	4	.250457E-02	4	.174033E-02
3	8	.760039E-01	4	.753252E-03
4	5	.132805E-01	4	.279643E-02
5	4	.716823E-02	4	.478021E-02
6	5	.356402E-02	4	.208882E-03
7	4	.354625E-03	3	.302951E-04
8	4	.211573E-02	3	.182664E-03
9	3	.604542E-03	2	.232164E-05
10	4	.119641E-01	4	.455832E-02
11	4	.130464E+00	4	.353892E-02
12	4	.129361E+00	4	.393477E-02
13	2	.128426E-05	1	.943582E-07
14	2	.831124E-06	1	.197656E-06
15	2	.193794E-05	1	.276215E-06
16	2	.130401E-05	1	.857718E-07
17	1	.456442E-08	1	.705316E-09
18	1	.343343E-07	1	.186947E-08
19	11	.103867E+00	5	.729780E-02
20	4	.188589E-02	3	.195450E-03
21	4	.112364E-02	3	.382116E-03
22	5	.115497E-01	4	.401051E-02
23	4	.610397E-02	4	.385898E-02
24	3	.542588E-04	3	.128426E-03

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $L^2(C_r)$ -space, for 24 IVP-s.



The numerical results of Table 2c are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$  for IVP's with equations 1-9 and 19-24 and with  $r=2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$  for IVP's with equations 10-18. The optimal method is performing worse on equation 24 compared to the usual method. But in larger interval, say,  $a = -1.9$ ,  $b = 1.9$  and  $r = 2.01$ ,  $n=38$ ,  $h=.1$ , it gives favorable results where also the number of iterations for convergence of solution for usual method is becoming high. The results are given below.

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
24:	31	.7966732E-02	4	.327223E-02

Overall, as compared to the usual method,  $\beta$ -optimal method is two decimal places better for IVP's with equations 3,9,11-13,16,19; one decimal place better for IVP's with equations 1,4,6-8,10,15,17, 18,20-22; and just better for IVP's with equations 2,5,14,23 and 24.

In Tables 3a, 3b and 3c we are presenting the numerical results for the  $\beta$ -optimal method corresponding to Stormer's 5-points usual method. From Table 3a it is seen that the optimal  $\hat{\beta}_{in}$ 's are depending on the nodal points and in a neighborhood of the origin, the optimal  $\hat{\beta}_{in}$ 's are close to the corresponding usual  $\beta_1$ 's. From table 3b, it is seen that at a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ . At the beginning of Table 3b,  $\|\hat{T}_n\|^2$  is much less than  $\|T_n\|^2$  compared to the end of this table, what is seen from the corresponding graphs also.

Table - 3a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	$h^2 \hat{\beta}_{3n}$	$h^2 \hat{\beta}_{4n}$
-1.3	.9957E-02	-.3300E-03	.3827E-03	-.5852E-04	.2790E-05
-1.2	.1027E-01	-.8884E-03	.7295E-03	-.1463E-03	.1027E-04
-1.1	.1054E-01	-.1436E-02	.1133E-02	-.2715E-03	.2395E-04
-1.0	.1077E-01	-.1959E-02	.1567E-02	-.4269E-03	.4400E-04
-.9	.1097E-01	-.2456E-02	.2021E-02	-.6068E-03	.7008E-04
-.8	.1115E-01	-.2930E-02	.2486E-02	-.8068E-03	.1018E-03
-.7	.1131E-01	-.3384E-02	.2959E-02	-.1024E-02	.1388E-03
-.6	.1146E-01	-.3818E-02	.3437E-02	-.1256E-02	.1806E-03
-.5	.1159E-01	-.4235E-02	.3917E-02	-.1500E-02	.2270E-03
-.4	.1172E-01	-.4637E-02	.4399E-02	-.1756E-02	.2778E-03
-.3	.1183E-01	-.5027E-02	.4883E-02	-.2023E-02	.3329E-03
-.2	.1194E-01	-.5408E-02	.5374E-02	-.2303E-02	.3927E-03
-.1	.1205E-01	-.5785E-02	.5873E-02	-.2597E-02	.4577E-03
.0	.1215E-01	-.6161E-02	.6388E-02	-.2910E-02	.5286E-03
.1	.1226E-01	-.6543E-02	.6923E-02	-.3244E-02	.6068E-03
.2	.1236E-01	-.6934E-02	.7488E-02	-.3606E-02	.6938E-03
.3	.1246E-01	-.7342E-02	.8092E-02	-.4003E-02	.7917E-03
.4	.1257E-01	-.7772E-02	.8746E-02	-.4445E-02	.9036E-03
.5	.1268E-01	-.8235E-02	.9468E-02	-.4945E-02	.1034E-02
.6	.1280E-01	-.8739E-02	.1028E-01	-.5521E-02	.1187E-02
.7	.1292E-01	-.9300E-02	.1120E-01	-.6198E-02	.1373E-02
.8	.1306E-01	-.9936E-02	.1228E-01	-.7011E-02	.1603E-02
.9	.1322E-01	-.1067E-01	.1357E-01	-.8014E-02	.1895E-02
1.0	.1340E-01	-.1154E-01	.1515E-01	-.9284E-02	.2277E-02
1.1	.1361E-01	-.1260E-01	.1714E-01	-.1095E-01	.2796E-02
1.2	.1386E-01	-.1393E-01	.1975E-01	-.1321E-01	.3530E-02
1.3	.1417E-01	-.1564E-01	.2328E-01	-.1643E-01	.4622E-02
1.4	.1457E-01	-.1795E-01	.2833E-01	-.2127E-01	.6348E-02
1.5	.1509E-01	-.2127E-01	.3605E-01	-.2915E-01	.9325E-02
1.6	.1583E-01	-.2642E-01	.4908E-01	-.4348E-01	.1512E-01
1.7	.1699E-01	-.3546E-01	.7451E-01	-.7422E-01	.2866E-01
1.8	.1909E-01	-.5482E-01	.1375E+00	-.1605E+00	.7109E-01

Table for optimal  $\beta$ 's of the  $\beta$ -optimal method in  $L^2(\hat{C}_r)$ -space corresponding to Stormer's 5-points usual method, at the nodal points, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=1.2458$ ,  $\beta_1=-.73333$ ,  $\beta_2=.80833$ ,  $\beta_3=-.4$ ,  $\beta_4=.079167$ .

Table - 3b

$x$	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.300	.258049E-03	.848220E-09
-1.200	.645653E-05	.123405E-09
-1.100	.356124E-06	.242352E-10
-1.000	.322349E-07	.615723E-11
-.900	.412492E-08	.194613E-11
-.800	.683616E-09	.740131E-12
-.700	.138693E-09	.329199E-12
-.600	.331516E-10	.167333E-12
-.500	.908783E-11	.954836E-13
-.400	.280310E-11	.603542E-13
-.300	.960427E-12	.419738E-13
-.200	.363022E-12	.318556E-13
-.100	.151519E-12	.262557E-13
.000	.708613E-13	.233946E-13
.100	.388029E-13	.225770E-13
.200	.270623E-13	.235053E-13
.300	.266105E-13	.265867E-13
.400	.370089E-13	.326048E-13
.500	.662970E-13	.434402E-13
.600	.139816E-12	.631345E-13
.700	.331589E-12	.100957E-12
.800	.868695E-12	.178863E-12
.900	.250916E-11	.354900E-12
1.000	.804176E-11	.800682E-12
1.100	.289479E-10	.209293E-11
1.200	.119190E-09	.650386E-11
1.300	.575983E-09	.249012E-10
1.400	.338795E-08	.123458E-09
1.500	.255763E-07	.851351E-09
1.600	.268741E-06	.909690E-08
1.700	.449456E-05	.179916E-06
1.800	.153874E-03	.919094E-05

Table for square norm of local truncation error functionals in  $L^2(C_r)$  space, for Stormer's 5-points usual and the corresponding  $\beta$ -optimal methods, at the nodal points with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ .

The results of Table 3c are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$  for IVP's with equations 1-9 and 19-24; and with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$  for IVP's with equations 10-18. As compared to usual method the optimal method is four decimal places better on equations 10, 11; three decimal places better on equations 3,6,8,9, 15,19; two decimal places better on equations

1,4,5,7,12-14,16-18,20-22 and one decimal place better on equations 2,23 and 24.

Table - 3c

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.304356E-01	.171511E-03
2	.632144E-02	.142098E-03
3	.167084E+00	.169000E-03
4	.289167E-01	.177737E-03
5	.186264E-01	.539045E-03
6	.748945E-02	.721988E-05
7	.854658E-03	.239009E-05
8	.515672E-02	.441774E-05
9	.163408E-02	.716423E-05
10	.402219E+02	.307110E-02
11	.670431E+02	.954700E+00
12	.670732E+02	.792798E+00
13	.635226E-03	.660198E-05
14	.266670E-03	.770471E-05
15	.152962E-02	.122524E-05
16	.700262E-03	.423805E-05
17	.148635E-05	.114005E-07
18	.672210E-05	.512201E-07
19	.246038E+00	.569904E-03
20	.415804E-02	.253098E-04
21	.282785E-02	.291817E-04
22	.253443E-01	.177735E-03
23	.161435E-01	.170197E-02
24	.151126E-03	.111921E-04

Table for number of iterations for convergence of solution and the average discretisation errors using Störmer's 5-points usual and the corresponding  $\beta$ -optimal methods in  $L^2(C_r)$ -space, for 24 IVP-s.

In  $L^2(\hat{C}_r)$  space, to determine the optimal coefficients  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$ , of an  $\alpha$ -optimal multistep method

$$(9) \quad \sum_{j=0}^k \hat{\alpha}_{j,n} y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0,$$

where  $\beta_j$ 's are prefixed according to some consistent and stable known usual method with highest degree polynomial precision. With reference to equation (13) of Chapter 1, we have the system of

normal equations given by

$$\hat{C} \hat{a} = \hat{d},$$

where

$$\hat{a} = \left( \hat{\alpha}_{0,n}, \dots, \hat{\alpha}_{k,n} \right)^T,$$

$$\hat{C}_{ij} = \frac{r^2}{\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_{n+k-i})^2}, \quad i, j = 0(1)k,$$

$$\text{and} \quad \hat{d}_i = - \frac{6r^2}{\pi} h^2 \sum_{j=\delta_{t_0}}^k \beta_j \frac{\bar{x}_{n+k-i}^2}{(r^2 - \bar{x}_{n+k-i} x_{n+k-j})^4}, \quad i = 0(1)k.$$

This system of equations can be solved for  $\hat{\alpha}_{j,n}$ ,  $j = 0(1)k$ .

The following theorem characterizes the  $\alpha$ -optimal multistep method (9) in  $L^2(\hat{C}_r)$  space.

Theorem 4: The optimal multistep method (9) in which  $\beta_j$ 's are prefixed and optimization is done with respect to  $\alpha_j$ 's in  $L^2(\hat{C}_r)$  space, is characterized by that it is locally interpolatory for functions

$$\left\{ y_i(x) = \frac{1}{(r^2 - x \bar{x}_{n+k-i})^2}, \quad i = 0(1)k \right\}.$$

Proof: The proof follows from Theorem 2 of Chapter 1 and using the definitions of  $K(z, \bar{t})$  and  $D_2(t, \bar{z})$  from (4) and (5).

Remark: If  $x_{n+k-i} = 0$ , for some  $i = 0(1)k$ , then the  $\alpha$ -optimal method (9) is locally interpolatory for constants, which in other words tells that  $\sum_{j=0}^k \hat{\alpha}_{j,n} = 0$ .

Table - 4a

$x_n$	$\hat{\alpha}_{0n}$	$\hat{\alpha}_{1n}$	$\hat{\alpha}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.5	-.1001E+01	.2003E+01	-.1002E+01	-.2471E-04	.1814E-08	.1500E-08
-1.4	-.1001E+01	.2001E+01	-.1001E+01	-.7565E-05	.2102E-09	.1806E-09
-1.3	-.1000E+01	.2001E+01	-.1000E+01	-.2567E-05	.3298E-10	.2934E-10
-1.2	-.1000E+01	.2000E+01	-.1000E+01	-.9309E-06	.6519E-11	.5973E-11
-1.1	-.1000E+01	.2000E+01	-.1000E+01	-.3514E-06	.1546E-11	.1452E-11
-1.0	-.1000E+01	.2000E+01	-.1000E+01	-.1349E-06	.4249E-12	.4072E-12
-.9	-.1000E+01	.2000E+01	-.1000E+01	-.5161E-07	.1322E-12	.1286E-12
-.8	-.1000E+01	.2000E+01	-.1000E+01	-.1924E-07	.4571E-13	.4498E-13
-.7	-.1000E+01	.2000E+01	-.1000E+01	-.6806E-08	.1738E-13	.1724E-13
-.6	-.1000E+01	.2000E+01	-.1000E+01	-.2208E-08	.7163E-14	.7133E-14
-.5	-.1000E+01	.2000E+01	-.1000E+01	-.6250E-09	.3187E-14	.3198E-14
-.4	-.1000E+01	.2000E+01	-.1000E+01	-.1422E-09	.1527E-14	.1520E-14
-.3	-.1000E+01	.2000E+01	-.1000E+01	-.2226E-10	.8361E-15	.8161E-15
-.2	-.1000E+01	.2000E+01	-.1000E+01	-.1637E-11	.5326E-15	.5298E-15
-.1	-.1000E+01	.2000E+01	-.1000E+01	-.4441E-15	.3779E-15	.3609E-15
.0	-.1000E+01	.2000E+01	-.1000E+01	.0000E+00	.3171E-15	.3250E-15
.1	-.1000E+01	.2000E+01	-.1000E+01	-.2220E-15	.3249E-15	.3231E-15
.2	-.1000E+01	.2000E+01	-.1000E+01	-.1642E-11	.4266E-15	.4329E-15
.3	-.1000E+01	.2000E+01	-.1000E+01	-.2226E-10	.7830E-15	.7775E-15
.4	-.1000E+01	.2000E+01	-.1000E+01	-.1422E-09	.1563E-14	.1553E-14
.5	-.1000E+01	.2000E+01	-.1000E+01	-.6250E-09	.3214E-14	.3196E-14
.6	-.1000E+01	.2000E+01	-.1000E+01	-.2208E-08	.7207E-14	.7181E-14
.7	-.1000E+01	.2000E+01	-.1000E+01	-.6806E-08	.1733E-13	.1721E-13
.8	-.1000E+01	.2000E+01	-.1000E+01	-.1924E-07	.4568E-13	.4495E-13
.9	-.1000E+01	.2000E+01	-.1000E+01	-.5161E-07	.1322E-12	.1287E-12
1.0	-.1000E+01	.2000E+01	-.1000E+01	-.1349E-06	.4249E-12	.4072E-12
1.1	-.1000E+01	.2000E+01	-.1000E+01	-.3514E-06	.1546E-11	.1452E-11
1.2	-.1000E+01	.2000E+01	-.1000E+01	-.9309E-06	.6519E-11	.5973E-11
1.3	-.1000E+01	.2001E+01	-.1000E+01	-.2567E-05	.3298E-10	.2934E-10
1.4	-.1001E+01	.2001E+01	-.1001E+01	-.7565E-05	.2102E-09	.1806E-09
1.5	-.1002E+01	.2003E+01	-.1001E+01	-.2471E-04	.1814E-08	.1500E-08

Table for optimal  $\alpha$ 's, their row-sum and the square norm of local truncation error functionals for Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods, in  $L^2(C_r)$ -space, at the nodal points, with  $r=2.1$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ; and  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ .

Table 4a shows that the optimal  $\hat{\alpha}_{in}$ 's are varying from point to point and they are very close to usual  $\alpha_i$ 's, at  $x=0$   $\hat{\alpha}_{in}$ 's are almost equal to  $\alpha_i$ 's;  $i=1,2,3$ . At a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ , except at  $x=0$ , where the errors are of the order of  $10^{-15}$ . Although the numerical calculations are done in double precision, such small

numbers are not so reliable. The reversal of the inequality at  $x = 0$ , in small numbers must be because of round-offs.

Table - 4b(i)

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.241691E-04	2	.106158E-04
2	2	.102218E-04	2	.186843E-05
3	2	.417599E-04	2	.304933E-04
4	2	.198576E-04	2	.114593E-04
5	2	.392783E-04	2	.744216E-04
6	2	.108795E-04	2	.726836E-05
7	2	.246490E-05	2	.141227E-05
8	2	.128880E-04	2	.247714E-05
9	2	.658772E-05	1	.150434E-06
10	1	.111844E-02	1	.877442E-03
11	1	.575402E-03	1	.205577E-03
12	1	.555353E-03	1	.190362E-03
13	2	.174323E-05	2	.490685E-06
14	2	.103607E-05	1	.129003E-06
15	2	.224814E-05	2	.154660E-05
16	2	.142465E-05	2	.837506E-06
17	1	.146848E-07	1	.114576E-08
18	1	.920340E-07	1	.158628E-06
19	2	.459516E-04	2	.130596E-04
20	2	.396036E-04	2	.612652E-05
21	2	.299086E-04	2	.190148E-04
22	2	.218887E-04	2	.186571E-04
23	2	.363654E-04	2	.105602E-03
24	2	.106490E-05	2	.170235E-04

Table - 4b(ii)

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
5	1	.139188E-05	1	.130059E-05
18	1	.183027E-07	1	.455087E-08
23	2	.425829E-05	2	.423013E-05
24	1	.146241E-05	1	.146155E-05

Tables for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual and the corresponding  $\alpha$ -optimal methods in  $L^2(C_r)$ -space, on 24 BVP-s.

The results of Table 4b(i) are obtained with  $r=2.1$ ,  $a=-1.6$ ,  $b=1.6$ ,  $n=32$ ,  $h=.1$  for BVP's with equations 1-9 and 19-24 and with  $r=2.8$ ,  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$  for equations 10-18. But still the

$\alpha$ -optimal method is producing worse results for BVP's with equations 5,18,23 and 24. Apparently it seems that, this situation happens because the singularities of the solutions of the differential equations 5,18,23 and 24 are not so pronounced. However with the same  $r$ ,  $r=2.01$ , but in smaller interval, viz.,  $a=-1$ ,  $b=1$  with  $n=20$ ,  $\alpha$ -optimal method gives just better result for equation 5, and with the same  $r$ ,  $r=2.01$ , but in more reduced interval viz.,  $a=-.6$ ,  $b=.6$  with  $n=12$ , the  $\alpha$ -optimal method gives just better results for equations 23 and 24. With  $r=2.8$ , in an interval  $a=-1.6$ ,  $b=1.6$  with  $n=32$ ,  $h=.1$   $\alpha$ -optimal method is producing better result on equation 18 and these results are shown in Table 4b(ii).

In  $L^2(\hat{C}_r)$  space, to determine the optimal coefficients  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k$  of  $\beta$ -optimal multistep method (7) subject to the condition

$$(10) \quad \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} = 1,$$

where  $\alpha_j$ 's are prefixed according to some consistent and stable known usual method (8) with highest degree polynomial precision, with reference to equation (15) of Chapter 1, we have the system of normal equations given by

$$\hat{C} \hat{b} = \hat{d},$$

where

$$\hat{b} = h^2 \left( \hat{\beta}_{\delta_{t_0},n}, \dots, \hat{\beta}_{k-1,n} \right)^T,$$



$$\begin{aligned}
\hat{C}_{ij} = & \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_{n+k-l})^4} \left[ 1 + \frac{8x_{n+k-j} \bar{x}_{n+k-l}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-l})} + \frac{10x_{n+k-j}^2 \bar{x}_{n+k-l}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-l})^2} \right] \\
& - \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_n)^4} \left[ 1 + \frac{8x_{n+k-j} \bar{x}_n}{(r^2 - x_{n+k-j} \bar{x}_n)} + \frac{10x_{n+k-j}^2 \bar{x}_n^2}{(r^2 - x_{n+k-j} \bar{x}_n)^2} \right] \\
& - \frac{12r^2}{\pi} \frac{1}{(r^2 - x_n \bar{x}_{n+k-l})^4} \left[ 1 + \frac{8x_n \bar{x}_{n+k-l}}{(r^2 - x_n \bar{x}_{n+k-l})} + \frac{10x_n^2 \bar{x}_{n+k-l}^2}{(r^2 - x_n \bar{x}_{n+k-l})^2} \right] \\
& + \frac{12r^2}{\pi} \frac{1}{(r^2 - x_n \bar{x}_n)^4} \left[ 1 + \frac{8x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{10x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right], \quad i, j = \delta_{t_0}(1)k-1,
\end{aligned}$$

and

$$\begin{aligned}
\hat{d}_1 = & - \frac{6r^2}{\pi} \sum_{j=0}^k \alpha_j \left\{ \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-l})^4} - \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_n)^4} \right\} \\
& - \frac{12r^2}{\pi} h^2 \left[ \frac{1}{(r^2 - x_n \bar{x}_{n+k-l})^4} \left\{ 1 + \frac{8x_n \bar{x}_{n+k-l}}{(r^2 - x_n \bar{x}_{n+k-l})} + \frac{10x_n^2 \bar{x}_{n+k-l}^2}{(r^2 - x_n \bar{x}_{n+k-l})^2} \right\} \right. \\
& \left. - \frac{1}{(r^2 - x_n \bar{x}_n)^4} \left\{ 1 + \frac{8x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{10x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right\} \right], \quad 1 = \delta_{t_0}(1)k-1.
\end{aligned}$$

This system of equations may be solved for  $h^2 \hat{\beta}_{j,n}$ ,  $j = \delta_{t_0}(1)k-1$ .

The following theorem characterizes the  $\beta$ -optimal multistep method (7) in  $L^2(\hat{C}_r)$  space subject to the condition (10).

**Theorem 5:** The optimal multistep method (7) in which  $\alpha_j$ 's are prefixed and  $\beta_j$ 's are optimized in  $L^2(\hat{C}_r)$  space, subject to the condition (10) is characterized by that it is locally interpolatory for functions

$$\left\{ y_1(x) = \frac{x^2}{(r^2 - x \bar{x}_{n+k-l})^4} - \frac{x^2}{(r^2 - x \bar{x}_n)^4}, \quad i = \delta_{t_0}(1)k-1 \right\}.$$

Proof: The proof follows from Theorem 3 of Chapter 1, and the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$  and  $D_2''(t, \bar{z})$  from (4), (5) and (6).

In  $L^2(\hat{C}_r)$  space, we have to determine the optimal coefficients  $\hat{\alpha}_{i,n}$ ,  $i = 0(1)k$ , and  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k-1$  of an  $(\alpha, \beta)$ -optimal multistep method

$$(11) \quad \sum_{j=0}^k \hat{\alpha}_{j,n} y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} f_{n+k-j} = 0,$$

subject to the condition (10). With reference to equation (18) and (19) of Chapter 1, we have the system of normal equations given by

$$\hat{C} \hat{b} = \hat{d},$$

where

$$\hat{b} = \left( \hat{\alpha}_{0,n}, \dots, \hat{\alpha}_{k,n}, h^2 \hat{\beta}_{\delta_{t_0},n}, \dots, h^2 \hat{\beta}_{k-1,n} \right)^T,$$

$$\hat{C}_{ij} = \frac{r^2}{\pi(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2}, \quad i, j = 0(1)k,$$

$$\hat{C}_{ij} = \frac{6r^2}{\pi} \left[ \frac{\bar{x}_{n+k-1}^2}{(r^2 - \bar{x}_{n+k-1} x_{n+k-j})^4} - \frac{\bar{x}_{n+k-1}^2}{(r^2 - \bar{x}_{n+k-1} x_n)^4} \right], \quad \begin{matrix} i=0(1)k, \\ j=k+1(1)2k+1-\delta_{t_0}, \end{matrix}$$

$$\hat{C}_{ij} = \frac{6r^2}{\pi} \left[ \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^4} - \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_n)^4} \right]; \quad \begin{matrix} i=k+1(1)2k+1-\delta_{t_0} \\ j=0(1)k \end{matrix}$$

$$\hat{C}_{ij} = \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^4} \left[ 1 + \frac{8x_{n+k-j} \bar{x}_{n+k-1}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})} + \frac{10x_{n+k-j}^2 \bar{x}_{n+k-1}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2} \right]$$

$$- \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_n)^4} \left[ 1 + \frac{8x_{n+k-j} \bar{x}_n}{(r^2 - x_{n+k-j} \bar{x}_n)} + \frac{10x_{n+k-j}^2 \bar{x}_n^2}{(r^2 - x_{n+k-j} \bar{x}_n)^2} \right]$$

$$\begin{aligned}
& - \frac{12r^2}{\pi} \frac{1}{(r^2 - x_n \bar{x}_{n+k-1})^4} \left[ 1 + \frac{8x_n \bar{x}_{n+k-1}}{(r^2 - x_n \bar{x}_{n+k-1})} + \frac{10x_n^2 \bar{x}_{n+k-1}^2}{(r^2 - x_n \bar{x}_{n+k-1})^2} \right] \\
& + \frac{12r^2}{\pi} \frac{1}{(r^2 - x_n \bar{x}_n)^4} \left[ 1 + \frac{8x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{10x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right], \quad i, j = k+1(1)2k+1-\delta_{t_0},
\end{aligned}$$

and

$$\hat{d}_i = h^2 \frac{6r^2}{\pi} \frac{\bar{x}_{n+k-1}^2}{(r^2 - \bar{x}_{n+k-1} x_n)^4}, \quad i = 0(1)k,$$

$$\begin{aligned}
\hat{d}_i &= - h^2 \frac{12r^2}{\pi} \left[ \frac{1}{(r^2 - x_n \bar{x}_{n+k-1})^4} \left\{ 1 + \frac{8x_n \bar{x}_{n+k-1}}{(r^2 - x_n \bar{x}_{n+k-1})} + \frac{10x_n^2 \bar{x}_{n+k-1}^2}{(r^2 - x_n \bar{x}_{n+k-1})^2} \right\} \right. \\
&\quad \left. - \frac{1}{(r^2 - x_n \bar{x}_n)^4} \left\{ 1 + \frac{8x_n \bar{x}_n}{(r^2 - x_n \bar{x}_n)} + \frac{10x_n^2 \bar{x}_n^2}{(r^2 - x_n \bar{x}_n)^2} \right\} \right], \quad i, j = k+1(1)2k+1-\delta_{t_0}.
\end{aligned}$$

The following theorem characterizes the  $(\alpha, \beta)$ -optimal method (11) in  $L^2(\hat{C}_r)$  space subject to the condition (10).

**Theorem 6:** The optimal multistep method (11) in which  $\alpha_j$ 's as well as  $\beta_j$ 's are optimized in  $L^2(\hat{C}_r)$  space, subject to the condition (10) is characterized by that it is locally interpolatory for functions  $\{y_{1i}(x), i=\delta_{t_0}(1)k-1\} \cup \{y_{2i}(x), i=0(1)k-1\}$ , where

$$y_{1i}(x) = \frac{x^2}{(r^2 - x \bar{x}_{n+k-1})^4} - \frac{x^2}{(r^2 - x \bar{x}_n)^4}, \quad i = \delta_{t_0}(1)k-1$$

$$\text{and } y_{2i}(x) = \frac{1}{(r^2 - x \bar{x}_{n+k-1})^2}, \quad i = 0(1)k-1.$$

**Proof:** The proof follows from Theorem 4 of Chapter 1, and the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$  and  $D_2''(t, \bar{z})$  from (4), (5) and (6).

Table - 5a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.6432E-03	.8980E-02	.3767E-03	.1000E-01	.2487E-04	.2920E-05
-1.6	.7176E-03	.8672E-02	.6109E-03	.1000E-01	.4108E-06	.5854E-07
-1.5	.7565E-03	.8540E-02	.7039E-03	.1000E-01	.1811E-07	.2997E-08
-1.4	.7793E-03	.8471E-02	.7500E-03	.1000E-01	.1452E-08	.2751E-09
-1.3	.7938E-03	.8430E-02	.7761E-03	.1000E-01	.1746E-09	.3779E-10
-1.2	.8036E-03	.8404E-02	.7923E-03	.1000E-01	.2820E-10	.6988E-11
-1.1	.8106E-03	.8386E-02	.8031E-03	.1000E-01	.5706E-11	.1626E-11
-1.0	.8157E-03	.8374E-02	.8107E-03	.1000E-01	.1380E-11	.4550E-12
-.9	.8197E-03	.8364E-02	.8162E-03	.1000E-01	.3865E-12	.1484E-12
-.8	.8228E-03	.8357E-02	.8205E-03	.1000E-01	.1223E-12	.5502E-13
-.7	.8254E-03	.8351E-02	.8238E-03	.1000E-01	.4305E-13	.2277E-13
-.6	.8275E-03	.8346E-02	.8265E-03	.1000E-01	.1664E-13	.1034E-13
-.5	.8293E-03	.8342E-02	.8287E-03	.1000E-01	.6986E-14	.5053E-14
-.4	.8307E-03	.8339E-02	.8304E-03	.1000E-01	.3262E-14	.2708E-14
-.3	.8319E-03	.8336E-02	.8317E-03	.1000E-01	.1700E-14	.1565E-14
-.2	.8327E-03	.8335E-02	.8326E-03	.1000E-01	.1011E-14	.9879E-15
-.1	.8331E-03	.8334E-02	.8331E-03	.1000E-01	.6583E-15	.6567E-15
.0	.8333E-03	.8333E-02	.8333E-03	.1000E-01	.5686E-15	.5686E-15
.1	.8331E-03	.8334E-02	.8331E-03	.1000E-01	.5920E-15	.5904E-15
.2	.8326E-03	.8335E-02	.8327E-03	.1000E-01	.9090E-15	.8863E-15
.3	.8317E-03	.8336E-02	.8319E-03	.1000E-01	.1656E-14	.1521E-14
.4	.8304E-03	.8339E-02	.8307E-03	.1000E-01	.3280E-14	.2725E-14
.5	.8287E-03	.8342E-02	.8293E-03	.1000E-01	.7039E-14	.5106E-14
.6	.8265E-03	.8346E-02	.8275E-03	.1000E-01	.1669E-13	.1039E-13
.7	.8238E-03	.8351E-02	.8254E-03	.1000E-01	.4311E-13	.2283E-13
.8	.8205E-03	.8357E-02	.8228E-03	.1000E-01	.1223E-12	.5497E-13
.9	.8162E-03	.8364E-02	.8197E-03	.1000E-01	.3864E-12	.1482E-12
1.0	.8107E-03	.8374E-02	.8157E-03	.1000E-01	.1380E-11	.4550E-12
1.1	.8031E-03	.8386E-02	.8106E-03	.1000E-01	.5706E-11	.1626E-11
1.2	.7923E-03	.8404E-02	.8036E-03	.1000E-01	.2820E-10	.6988E-11
1.3	.7761E-03	.8430E-02	.7938E-03	.1000E-01	.1746E-09	.3779E-10
1.4	.7500E-03	.8471E-02	.7793E-03	.1000E-01	.1452E-08	.2751E-09
1.5	.7039E-03	.8540E-02	.7565E-03	.1000E-01	.1811E-07	.2997E-08
1.6	.6109E-03	.8672E-02	.7176E-03	.1000E-01	.4108E-06	.5854E-07
1.7	.3767E-03	.8980E-02	.6432E-03	.1000E-01	.2487E-04	.2920E-05

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals for Cowell's 3-point usual method and the corresponding  $\beta$ -optimal method with restriction, in  $L^2(C_r)$ -space, at the nodal points, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ; and  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ .

From Table 5a it is seen that the optimal  $\hat{\beta}_{1n}$ 's are depending on the nodal points and at a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ . In a neighborhood of the origin the optimal  $\hat{\beta}_{1n}$ 's are close to the

corresponding usual  $\beta_i$ 's. At  $x=0$ ,  $\hat{\beta}_{in}$  and  $\beta_i$ ,  $i=1,2,3$ , as well as  $\|\hat{T}_n\|^2$  and  $\|T_n\|^2$  are numerically equal.

Table - 5b

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.813027E-04
2	2	.610703E-04	2	.499576E-04
3	3	.430195E-03	2	.109803E-03
4	2	.133292E-03	2	.603006E-04
5	2	.348263E-03	2	.199561E-03
6	3	.127772E-03	2	.321776E-04
7	2	.313175E-04	2	.797586E-05
8	2	.133523E-03	2	.446397E-04
9	2	.735570E-04	2	.246009E-04
10	1	.170782E-02	1	.190608E-03
11	1	.738872E-03	1	.298466E-03
12	1	.718937E-03	1	.288824E-03
13	2	.298705E-05	2	.116579E-05
14	2	.167071E-05	2	.728604E-06
15	2	.392662E-05	2	.137844E-05
16	2	.236517E-05	2	.827569E-06
17	1	.257920E-07	1	.948448E-08
18	1	.151826E-06	1	.700690E-07
19	3	.509406E-03	2	.133109E-03
20	3	.464907E-03	2	.125596E-03
21	2	.304134E-03	2	.116677E-03
22	2	.208304E-03	2	.799850E-04
23	2	.300455E-03	2	.163980E-03
24	2	.287372E-04	2	.161187E-04

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method with restriction in  $L^2(C_r)$ -space, for 24 BVP-s.

Results in Table 5b are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$  for BVP's with equations 1-9 and 19-24 and with  $r=2.8$ ,  $a=-2.1$ ,  $b=2.1$ ,  $n=42$ ,  $h=.1$  for BVP's with equations 10-18. Table 5b reveals that as compared to usual method  $\beta$ -optimal method with restriction for BVP is one decimal place better on equations 1,4,6-8,10,14,16-18,22; and just better on equations 2,3,5,9, 11-13,15,19-21,23 and 24.

Table - 5c

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.485217E-02
2	4	.250457E-02	4	.173155E-02
3	8	.760039E-01	5	.173120E-01
4	5	.132805E-01	4	.509940E-02
5	4	.716823E-02	4	.423506E-02
6	5	.356402E-02	4	.963058E-03
7	4	.354625E-03	3	.100186E-03
8	4	.211573E-02	4	.750653E-03
9	3	.604542E-03	3	.220553E-03
10	4	.119641E-01	4	.513783E-02
11	4	.130464E+00	4	.753139E-02
12	4	.129361E+00	4	.746002E-02
13	2	.128426E-05	2	.551296E-06
14	2	.831124E-06	1	.360443E-06
15	2	.193794E-05	2	.805851E-06
16	2	.130401E-05	2	.530703E-06
17	1	.456442E-08	1	.398993E-09
18	1	.343343E-07	1	.277793E-08
19	11	.103867E+00	6	.200835E-01
20	4	.188589E-02	3	.484536E-03
21	4	.112364E-02	3	.397635E-03
22	5	.115497E-01	4	.409179E-02
23	4	.610397E-02	4	.357735E-02
24	3	.542588E-04	2	.685449E-05

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method with restriction in  $L^2(C_r)$ -space, for 24 IVP-s.

The results of Table 5c are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$  and  $h=.1$  for IVP's with equations 1-9 and 19-24 and with  $r=2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$  for IVP's with equations

10-18. Table 5c shows that as compared to usual method  $\beta$ -optimal method with restriction for IVP is two decimal places better on equations 11,12; one decimal place better on equations 1,4,6,8, 10, 12,15-22,24; and just better on equations 2,3,5,7,14 and 23.

#### 5.4 Optimal multistep methods in $L^2(\hat{C}_r)$ -space Interpolatory For Polynomials.

In section 4.4, we have discussed optimal multistep methods, in  $H^2(C_r)$  space, interpolatory for polynomials of a certain degree. Similar to that methodology,  $\beta$ -optimal multistep method in  $L^2(\hat{C}_r)$  space, interpolatory for polynomials of degree  $q < k + \delta_{t_1} + 1$ , is given by

$$(12) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} + h^2 \sum_{j=0}^{q-2} \gamma_j^u \nabla^j f(x_{n+k-\delta_{t_0}}, Y_{n+k-\delta_{t_0}}) = \\ - h^2 \sum_{j=q-1}^{k-\delta_{t_0}} \hat{\gamma}_{j,n} \nabla^j f(x_{n+k-\delta_{t_0}}, Y_{n+k-\delta_{t_0}}),$$

with prefixed  $\alpha_i$ 's such that  $\sum_{i=0}^k \alpha_i = 0$ ,  $\sum_{i=0}^k i \alpha_i = 0$ , and with prefixed  $\gamma_j^u$ ,  $j = 0(1)q-2$ , as in the corresponding usual method. The optimal coefficients  $\hat{\gamma}_{j,n}$ ,  $j = q-1(1)k-\delta_{t_0}$ , are to satisfy the system of normal equations given by

$$\hat{C} \hat{b} = \hat{d}^1$$

where

$$\hat{b} = \left( \hat{\gamma}_{q-1,n}, \hat{\gamma}_{q,n}, \dots, \hat{\gamma}_{k-\delta_{t_0},n} \right)^T,$$

$$\hat{C}_{ij} = h^2 \sum_{m=0}^j \sum_{l=0}^i (-1)^{l+m-1} C_{i-1}^l C_m^j \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}-1})^4}.$$

$$\cdot \left[ 1 + \frac{8x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}}^{-1}}{(r^2 - x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}}^{-1})} + \frac{10x_{n+k-\delta_{t_0}}^2 \bar{x}_{n+k-\delta_{t_0}}^2}{(r^2 - x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}}^{-1})^2} \right],$$

and

$$\begin{aligned} \hat{d}_1 = & - \sum_{j=0}^k \alpha_j \sum_{l=0}^1 (-1)^{1-l} C_1 \frac{6r^2}{\pi} \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-\delta_{t_0}}^{-1})^4} \\ & - h^2 \sum_{j=0}^{q-2} \gamma_j^u \sum_{m=0}^j \sum_{l=0}^1 (-1)^{1+m-l} C_1^j C_m \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}}^{-1})^4} \cdot \\ & \cdot \left[ 1 + \frac{8x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}}^{-1}}{(r^2 - x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}}^{-1})} + \frac{10x_{n+k-\delta_{t_0}}^2 \bar{x}_{n+k-\delta_{t_0}}^2}{(r^2 - x_{n+k-\delta_{t_0}}^{-m} \bar{x}_{n+k-\delta_{t_0}}^{-1})^2} \right] \\ & i = q-1(1)k-\delta_{t_0}. \end{aligned}$$

Thus we get the following theorem.

**Theorem 7:** The optimal multistep method (12) interpolatory for polynomials of degree  $q$  is characterized by that it is locally interpolatory for functions

$$\{x^i; i = 1(1)q\} \cup \{h_1(x); i = q-1(1)k-\delta_{t_0}\},$$

$$\text{where } h_1(x) = \sum_{l=0}^1 (-1)^{1-l} C_1 \frac{x^2}{(r^2 - x \bar{x}_{n+k-\delta_{t_0}}^{-1})^4}.$$

In the following Tables 6a-6c we are presenting the numerical results for  $\beta$ -optimal method interpolatory for polynomials of degree 3 corresponding to Cowell's 3-points usual method. From Table 6a, it is clear that the optimal  $\hat{\beta}_{in}$ 's are depending on nodal points and at a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ . In a neighborhood of the origin, the optimal  $\hat{\beta}_{in}$ 's are close to the corresponding usual  $\beta_i$ 's;  $i=0,1,2$ ; at  $x=0$ , they as well as  $\|\hat{T}_n\|^2$  and  $\|T_n\|^2$  are



numerically the same .

Table - 6a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.7126E-03	.8575E-02	.7126E-03	.1000E-01	.2487E-04	.6813E-05
-1.6	.7664E-03	.8467E-02	.7664E-03	.1000E-01	.4108E-06	.1306E-06
-1.5	.7911E-03	.8418E-02	.7911E-03	.1000E-01	.1811E-07	.6347E-08
-1.4	.8045E-03	.8391E-02	.8045E-03	.1000E-01	.1452E-08	.5509E-09
-1.3	.8126E-03	.8375E-02	.8126E-03	.1000E-01	.1746E-09	.7131E-10
-1.2	.8178E-03	.8364E-02	.8178E-03	.1000E-01	.2820E-10	.1238E-10
-1.1	.8214E-03	.8357E-02	.8214E-03	.1000E-01	.5706E-11	.2696E-11
-1.0	.8240E-03	.8352E-02	.8240E-03	.1000E-01	.1380E-11	.7036E-12
-.9	.8260E-03	.8348E-02	.8260E-03	.1000E-01	.3865E-12	.2133E-12
-.8	.8275E-03	.8345E-02	.8275E-03	.1000E-01	.1223E-12	.7338E-13
-.7	.8287E-03	.8343E-02	.8287E-03	.1000E-01	.4305E-13	.2818E-13
-.6	.8297E-03	.8341E-02	.8297E-03	.1000E-01	.1664E-13	.1193E-13
-.5	.8306E-03	.8339E-02	.8306E-03	.1000E-01	.6986E-14	.5487E-14
-.4	.8314E-03	.8337E-02	.8314E-03	.1000E-01	.3262E-14	.2809E-14
-.3	.8321E-03	.8336E-02	.8321E-03	.1000E-01	.1700E-14	.1582E-14
-.2	.8327E-03	.8335E-02	.8327E-03	.1000E-01	.1011E-14	.9892E-15
-.1	.8331E-03	.8334E-02	.8331E-03	.1000E-01	.6583E-15	.6567E-15
.0	.8333E-03	.8333E-02	.8333E-03	.1000E-01	.5686E-15	.5686E-15
.1	.8331E-03	.8334E-02	.8331E-03	.1000E-01	.5920E-15	.5904E-15
.2	.8327E-03	.8335E-02	.8327E-03	.1000E-01	.9090E-15	.8876E-15
.3	.8321E-03	.8336E-02	.8321E-03	.1000E-01	.1656E-14	.1538E-14
.4	.8314E-03	.8337E-02	.8314E-03	.1000E-01	.3280E-14	.2826E-14
.5	.8306E-03	.8339E-02	.8306E-03	.1000E-01	.7039E-14	.5540E-14
.6	.8297E-03	.8341E-02	.8297E-03	.1000E-01	.1669E-13	.1198E-13
.7	.8287E-03	.8343E-02	.8287E-03	.1000E-01	.4311E-13	.2824E-13
.8	.8275E-03	.8345E-02	.8275E-03	.1000E-01	.1223E-12	.7334E-13
.9	.8260E-03	.8348E-02	.8260E-03	.1000E-01	.3864E-12	.2132E-12
1.0	.8240E-03	.8352E-02	.8240E-03	.1000E-01	.1380E-11	.7036E-12
1.1	.8214E-03	.8357E-02	.8214E-03	.1000E-01	.5706E-11	.2696E-11
1.2	.8178E-03	.8364E-02	.8178E-03	.1000E-01	.2820E-10	.1238E-10
1.3	.8126E-03	.8375E-02	.8126E-03	.1000E-01	.1746E-09	.7131E-10
1.4	.8045E-03	.8391E-02	.8045E-03	.1000E-01	.1452E-08	.5509E-09
1.5	.7911E-03	.8418E-02	.7911E-03	.1000E-01	.1811E-07	.6347E-08
1.6	.7664E-03	.8467E-02	.7664E-03	.1000E-01	.4108E-06	.1306E-06
1.7	.7126E-03	.8575E-02	.7126E-03	.1000E-01	.2487E-04	.6813E-05

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals, in  $L^2(C_r)$  space, for Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for polynomials of degree 3, at the nodal points, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ;  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ ;  $\gamma_0=1$ ,  $\gamma_1=-1$ ,  $\gamma_2=1/12$ .

Table - 6b

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.240920E-04
2	2	.610703E-04	2	.461772E-04
3	3	.430195E-03	2	.121004E-04
4	2	.133292E-03	2	.642565E-04
5	2	.348263E-03	2	.120095E-03
6	3	.127772E-03	2	.101242E-04
7	2	.313175E-04	2	.245537E-05
8	2	.133523E-03	2	.124323E-04
9	2	.735570E-04	2	.670919E-05
10	1	.111844E-02	1	.279828E-03
11	1	.575402E-03	1	.505366E-03
12	1	.555353E-03	1	.503450E-03
13	2	.174323E-05	1	.234601E-06
14	2	.103607E-05	1	.312393E-07
15	2	.224814E-05	2	.561019E-06
16	2	.142465E-05	1	.199793E-06
17	1	.146848E-07	1	.143021E-08
18	1	.920340E-07	1	.154712E-07
19	3	.509406E-03	2	.175912E-04
20	3	.464907E-03	2	.150172E-04
21	2	.304134E-03	2	.432583E-04
22	2	.208304E-03	2	.307457E-04
23	2	.300455E-03	2	.132747E-03
24	2	.287372E-04	2	.102816E-05

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for polynomials of degree 3, in  $L^2(C_r)$  space, for 24 BVP-s.

The numerical results of Table 6b are obtained for BVP's with equations 1-9 and 19-24 with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ; and for BVP's with equations 10-18 with  $r=2.8$ ,  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ . Table 6b reveals that as compared to usual method  $\beta$ -optimal method interpolatory for polynomials of degree 3 for BVP is two decimal places better on equation 14; one decimal place better on equations 1,3,4,6-10,13,15-17,19-22 and 24; and just better on equations 2,5,11,12,18 and 23.

Table - 6c

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.105817E-02
2	4	.250457E-02	4	.154352E-02
3	8	.760039E-01	4	.902492E-03
4	5	.132805E-01	4	.450808E-02
5	4	.716823E-02	4	.233608E-02
6	5	.356402E-02	4	.353335E-03
7	4	.354625E-03	3	.357746E-04
8	4	.211573E-02	3	.150320E-03
9	3	.604542E-03	3	.405464E-04
10	4	.119641E-01	4	.625978E-02
11	4	.130464E+00	4	.830081E-02
12	4	.129361E+00	4	.893553E-02
13	2	.128426E-05	1	.145172E-06
14	2	.831124E-06	1	.182743E-07
15	2	.193794E-05	2	.456883E-06
16	2	.130401E-05	1	.213949E-06
17	1	.456442E-08	1	.264896E-09
18	1	.343343E-07	1	.107190E-07
19	11	.103867E+00	5	.436396E-02
20	4	.188589E-02	3	.112072E-03
21	4	.112364E-02	3	.129308E-03
22	5	.115497E-01	4	.139534E-02
23	4	.610397E-02	4	.245008E-02
24	3	.542588E-04	2	.422862E-05

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for polynomials of degree 3, in  $L^2(C_r)$  space, for 24 IVP-s.

The results of Table 6c are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$  for IVP's with equations 1-9 and 19-24; and with  $r = 2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$  for IVP's with equations 10-18. Table 6c reveals that as compared to usual method,  $\beta$ -optimal method interpolatory for polynomials of degree 3 for IVP is two decimal places better on equation 3,11,12,19; one decimal place better on equations 1,4,6-10,13-17,20-22 and 24; and just better on equations 2,5,18 and 23.

Similar to the  $\alpha$ -optimal multistep method in  $H^2(C_r)$  space, interpolatory for polynomials of degree  $q$ , discussed in section 4.4, the  $\alpha$ -optimal multistep method in  $L^2(\hat{C}_r)$  space,

$$(13) \quad \sum_{j=2}^q \gamma_j^u \nabla^j y_{n+k} + \sum_{j=q+1}^k \gamma_j \nabla^j y_{n+k} + h^2 \sum_{j=\delta_{t0}}^k \beta_j f_{n+k-j} = 0,$$

with prefixed  $\beta_j$ 's and  $\gamma_j^u$ 's as in the corresponding usual method are to satisfy the system of normal equations

$$\hat{C} \hat{a} = \hat{d},$$

where

$$\hat{a} = \left( \hat{\gamma}_{q+1,n}, \dots, \hat{\gamma}_{k,n} \right)^T,$$

$$\hat{C}_{ij} = \sum_{m=0}^j \sum_{l=0}^i (-1)^{1+m} {}^1C_1 {}^jC_m \frac{r^2}{\pi} \frac{1}{(r^2 - x_{n+k-m} \bar{x}_{n+k-l})^2}, \quad i, j = q+1(1)k$$

$$\begin{aligned} \hat{d}_i &= - \sum_{j=2}^q \gamma_j^u \sum_{m=0}^j \sum_{l=0}^i (-1)^{1+m} {}^1C_1 {}^jC_m \frac{r^2}{\pi} \frac{1}{(r^2 - x_{n+k-m} \bar{x}_{n+k-l})^2} \\ &\quad - h^2 \sum_{j=\delta_{t0}}^k \beta_j \sum_{l=0}^i (-1)^{1+l} {}^1C_1 \frac{6r^2}{\pi} \frac{\bar{x}_{n+k-l}^2}{(r^2 - \bar{x}_{n+k-l} x_{n+k-j})^4}, \quad i = q+1(1)k. \end{aligned}$$

Hence we get the following theorem.

**Theorem 8 :** The optimal multistep method (13) interpolatory for polynomials of degree  $q$  is characterized by that it is locally interpolatory for functions

$$\{x^i; i = 1(1)q\} \cup \{\underline{h}_i(x); i = q+1(1)k\}$$

$$\text{where } h_i(x) = \sum_{l=0}^i (-1)^{1+l} {}^1C_1 \frac{1}{(r^2 - x \bar{x}_{n+k-l})^2}, \quad i = q+1(1)k.$$

### 5.5 Optimal Multistep Methods in $L^2(\hat{C}_r)$ -Space Interpolatory for a Set of Preassigned Functions.

To obtain the coefficients  $\hat{\beta}_{j,n}^F$ ,  $j = \delta_{t_0}(1)k$  of a  $\beta$ -optimal multistep method

$$(14) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n}^F f_{n+k-j} = 0,$$

in  $L^2(\hat{C}_r)$  space with prefixed  $\alpha_j$ 's, interpolatory for  $q$  number of linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$ , we have to solve a system of linear equations given in the matrix form as

$$\begin{bmatrix} A & F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} b \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where  $A = (A_{ij})$ ,  $i, j = \delta_{t_0}(1)k$ , with

$$A_{ij} = \frac{12r^2}{\pi} \frac{1}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^4} \cdot \left[ 1 + \frac{8x_{n+k-j} \bar{x}_{n+k-1}}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})} + \frac{10x_{n+k-j}^2 \bar{x}_{n+k-1}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^2} \right], \quad i, j = \delta_{t_0}(1)k,$$

$$F = (F_{ij})_{\substack{i=1(1)q \\ j=\delta_{t_0}(1)k}}, \quad \text{with } F_{ij} = f_i''(x_{n+k-j}),$$

$$b = h^2 \left( \hat{\beta}_{\delta_{t_0}, n}^F, \hat{\beta}_{\delta_{t_0}+1, n}^F, \dots, \hat{\beta}_{k, n}^F \right)^T, \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T,$$

$$c = (c_{\delta_{t_0}}, c_{\delta_{t_0}+1}, \dots, c_k)^T, \quad \text{with } c_i = - \sum_{j=0}^k \alpha_j \frac{6r^2}{\pi} \frac{x_{n+k-j}^2}{(r^2 - x_{n+k-j} \bar{x}_{n+k-1})^4},$$

and

$$f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T, \quad \text{with } \tilde{f}_i = - \sum_{j=0}^k \alpha_j f_i(x_{n+k-j}).$$

This method is characterized as follows.

**Theorem 9:** The  $\beta$ -optimal multistep method (14), in  $L^2(\hat{C}_r)$  space, interpolatory for linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$  is characterized by that, it is locally interpolatory for the functions

$$\{f_1, f_2, \dots, f_q\} \cup \{h_i, i = \delta_{t_0}(1)k - q\},$$

where

$$h_i(x) = \frac{x^2}{(r^2 - x\bar{x}_{n+k-1})^4} - \sum_{j=k-q+1}^k \bar{g}_{j+q-k, i+1-\delta_{t_0}} \frac{x^2}{(r^2 - x\bar{x}_{n+k-j})^4}$$

and

$$G = P^{-1}E = (g_{ij})_{\substack{i=1(1)q \\ j=1(1)k-q-\delta_{t_0}+1}},$$

where

$$P = \left( f''_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=k-q+1(1)k}} \quad \text{and} \quad E = \left( f''_i(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=\delta_{t_0}(1)k-q}}.$$

**Proof:** The proof follows from Theorem 8 of Chapter 1.

In the Tables 7a-7c we are presenting the numerical results for the  $\beta$ -optimal method interpolatory for linearly independent functions  $\exp(1.6x)$  and  $\exp(-1.6x)$ . From Table 7a it is clear that the optimal  $\hat{\beta}_{in}$ 's are depending on the nodal points, and in a neighborhood of the origin the optimal  $\hat{\beta}_{in}$ 's are close to the corresponding usual  $\beta_i$ 's. At a general point  $x_n$ ,  $\|\hat{T}_n\|^2 \leq \|T_n\|^2$ , except at  $x = -.1, 0$  and  $.1$  which must be because of round-offs in the middle of the table.

Table - 7a

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.7	.7110E-03	.8581E-02	.7110E-03	.1000E-01	.2487E-04	.6670E-05
-1.6	.7646E-03	.8473E-02	.7646E-03	.1000E-01	.4108E-06	.1262E-06
-1.5	.7893E-03	.8423E-02	.7893E-03	.1000E-01	.1811E-07	.6040E-08
-1.4	.8027E-03	.8395E-02	.8027E-03	.1000E-01	.1452E-08	.5163E-09
-1.3	.8108E-03	.8379E-02	.8108E-03	.1000E-01	.1746E-09	.6589E-10
-1.2	.8161E-03	.8368E-02	.8161E-03	.1000E-01	.2820E-10	.1132E-10
-1.1	.8199E-03	.8361E-02	.8199E-03	.1000E-01	.5706E-11	.2449E-11
-1.0	.8226E-03	.8355E-02	.8226E-03	.1000E-01	.1380E-11	.6397E-12
-.9	.8248E-03	.8351E-02	.8248E-03	.1000E-01	.3865E-12	.1956E-12
-.8	.8266E-03	.8347E-02	.8266E-03	.1000E-01	.1223E-12	.6838E-13
-.7	.8281E-03	.8344E-02	.8281E-03	.1000E-01	.4305E-13	.2681E-13
-.6	.8294E-03	.8341E-02	.8294E-03	.1000E-01	.1664E-13	.1159E-13
-.5	.8305E-03	.8339E-02	.8305E-03	.1000E-01	.6986E-14	.5419E-14
-.4	.8314E-03	.8337E-02	.8314E-03	.1000E-01	.3262E-14	.2798E-14
-.3	.8321E-03	.8336E-02	.8321E-03	.1000E-01	.1700E-14	.1581E-14
-.2	.8326E-03	.8335E-02	.8326E-03	.1000E-01	.1011E-14	.9934E-15
-.1	.8329E-03	.8334E-02	.8329E-03	.1000E-01	.6583E-15	.6677E-15
.0	.8330E-03	.8334E-02	.8330E-03	.1000E-01	.5686E-15	.5828E-15
.1	.8329E-03	.8334E-02	.8329E-03	.1000E-01	.5920E-15	.6014E-15
.2	.8326E-03	.8335E-02	.8326E-03	.1000E-01	.9090E-15	.8918E-15
.3	.8321E-03	.8336E-02	.8321E-03	.1000E-01	.1656E-14	.1536E-14
.4	.8314E-03	.8337E-02	.8314E-03	.1000E-01	.3280E-14	.2815E-14
.5	.8305E-03	.8339E-02	.8305E-03	.1000E-01	.7039E-14	.5472E-14
.6	.8294E-03	.8341E-02	.8294E-03	.1000E-01	.1669E-13	.1164E-13
.7	.8281E-03	.8344E-02	.8281E-03	.1000E-01	.4311E-13	.2688E-13
.8	.8266E-03	.8347E-02	.8266E-03	.1000E-01	.1223E-12	.6834E-13
.9	.8248E-03	.8351E-02	.8248E-03	.1000E-01	.3864E-12	.1955E-12
1.0	.8226E-03	.8355E-02	.8226E-03	.1000E-01	.1380E-11	.6397E-12
1.1	.8199E-03	.8361E-02	.8199E-03	.1000E-01	.5706E-11	.2449E-11
1.2	.8161E-03	.8368E-02	.8161E-03	.1000E-01	.2820E-10	.1131E-10
1.3	.8108E-03	.8379E-02	.8108E-03	.1000E-01	.1746E-09	.6589E-10
1.4	.8027E-03	.8395E-02	.8027E-03	.1000E-01	.1452E-08	.5163E-09
1.5	.7893E-03	.8423E-02	.7893E-03	.1000E-01	.1811E-07	.6040E-08
1.6	.7646E-03	.8473E-02	.7646E-03	.1000E-01	.4108E-06	.1262E-06
1.7	.7110E-03	.8581E-02	.7110E-03	.1000E-01	.2487E-04	.6670E-05

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals in  $L^2(Cr)$  space for Cowell's 3-points usual method and the corresponding optimal method interpolatory for  $\exp(\pm 1.6x)$ , at the nodal points, with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$ ;  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ;  $\beta_0=.083333$ ,  $\beta_1=.833333$ ,  $\beta_2=.083333$ .

Table - 7b

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	2	.244871E-03	2	.198267E-04
2	2	.610703E-04	2	.413152E-04
3	3	.430195E-03	2	.865679E-05
4	2	.133292E-03	2	.594996E-04
5	2	.348263E-03	2	.101422E-03
6	3	.127772E-03	2	.106507E-04
7	2	.313175E-04	2	.256370E-05
8	2	.133523E-03	2	.102367E-04
9	2	.735570E-04	2	.570366E-05
10	1	.111844E-02	1	.236393E-03
11	1	.575402E-03	1	.392509E-03
12	1	.555353E-03	1	.389258E-03
13	2	.174323E-05	2	.496382E-06
14	2	.103607E-05	1	.239882E-06
15	2	.224814E-05	2	.699675E-06
16	2	.142465E-05	2	.354305E-06
17	1	.146848E-07	1	.220510E-08
18	1	.920340E-07	1	.373346E-08
19	3	.509406E-03	2	.229013E-04
20	3	.464907E-03	2	.206231E-04
21	2	.304134E-03	2	.345624E-04
22	2	.208304E-03	2	.249211E-04
23	2	.300455E-03	2	.115582E-03
24	2	.287372E-04	2	.273774E-05

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for functions  $\exp(\pm 1.6x)$ , in  $L^2(C_r)$  space, for 24 BVP-s.

The results of Table 7b are obtained with  $r=2.01$ ,  $a=-1.8$ ,  $b=1.8$ ,  $n=36$ ,  $h=.1$  for BVP's with equations 1-9 and 19-24 and with  $r=2.8$ ,  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$  for BVP's with equations 10-18. Table 7b reveals that as compared to usual method,  $\beta$ -optimal method interpolatory for linearly independent functions  $\exp(\pm 1.6x)$  for BVP's is two decimal places better on equation 3; one decimal place better on equations 1,4,6-10,14-22,24; and just better on equations 2,5,11-13,23.



Table - 7c

eqn.	itu	$\bar{e}_u$	ito	$\bar{e}_o$
1	5	.140932E-01	4	.757542E-03
2	4	.250457E-02	4	.133628E-02
3	8	.760039E-01	4	.147005E-02
4	5	.132805E-01	4	.404481E-02
5	4	.716823E-02	4	.185820E-02
6	5	.356402E-02	4	.372113E-03
7	4	.354625E-03	3	.375720E-04
8	4	.211573E-02	3	.104763E-03
9	3	.604542E-03	3	.289515E-04
10	4	.119641E-01	4	.596451E-02
11	4	.130464E+00	3	.244210E-03
12	4	.129361E+00	3	.839701E-03
13	2	.128426E-05	2	.652674E-06
14	2	.831124E-06	2	.446034E-06
15	2	.193794E-05	2	.875456E-06
16	2	.130401E-05	2	.576796E-06
17	1	.456442E-08	1	.212508E-09
18	1	.343343E-07	1	.876228E-08
19	11	.103867E+00	5	.514932E-02
20	4	.188589E-02	3	.132981E-03
21	4	.112364E-02	3	.934208E-04
22	5	.115497E-01	4	.103295E-02
23	4	.610397E-02	4	.203661E-02
24	3	.542588E-04	2	.174337E-05

Table for number of iterations for convergence of the solution and the average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for functions  $\exp(\pm 1.6x)$ , in  $L^2(C_r)$  space, for 24 IVP-s.

The results of Table 7c are obtained with  $r=2.01$ ,  $a=-1.7$ ,  $b=1.7$ ,  $n=34$ ,  $h=.1$  for IVP's with equations 1-9 and 19-24 and with  $r=2.7$ ,  $a=-1.4$ ,  $b=1.4$ ,  $n=28$ ,  $h=.1$  for IVP's with equations 10-18. Table 7c reveals that as compared to usual method  $\beta$ -optimal method interpolatory for linearly independent functions  $\exp(\pm 1.6x)$  for IVP is three decimal places better on equations 11,12; two decimal places better on equations 1,19,21; one decimal place better on equations 3,4,6-10,13,15-18,20,22 and 24; and just better on equations 2,5,14 and 23.

In Figure 1, we are showing the graphs for the square norm of local truncation error functionals for the usual Cowell's method with function evaluation at three points [Usual] and those for the corresponding  $\beta$ -optimal method [Beta-optimal(1)],  $\beta$ -optimal method with restriction [Beta-Optimal(2)],  $\beta$ -optimal method interpolatory for polynomials of degree 3 [Beta-optimal(3)] and  $\beta$ -optimal method interpolatory for linearly independent functions  $\exp(\pm 1.6x)$  [Beta-optimal(4)], with the numerical results given in the Tables 2a, 5a, 6a and 7a. From Figure 1, we see that, the curves for Beta-Optimal(2), Beta-Optimal(3), Beta-Optimal(4) lie in between the curves for Usual and Beta-Optimal(1). So,  $\beta$ -optimal method is giving more optimized results than other optimal methods. Figure 2 presents the graphs for the square norm of local truncation error functionals for the usual Stormer's method with function evaluation at five points [Usual] and that for the corresponding  $\beta$ -optimal method [Beta-optimal(5)]. Figure 3 presents the square norm of local truncation error functional for the usual Cowell's method with function evaluation at three points [Usual] and that for the corresponding  $\alpha$ -optimal method. This figure reveals that  $\alpha$ -optimal method is not giving so promising results.

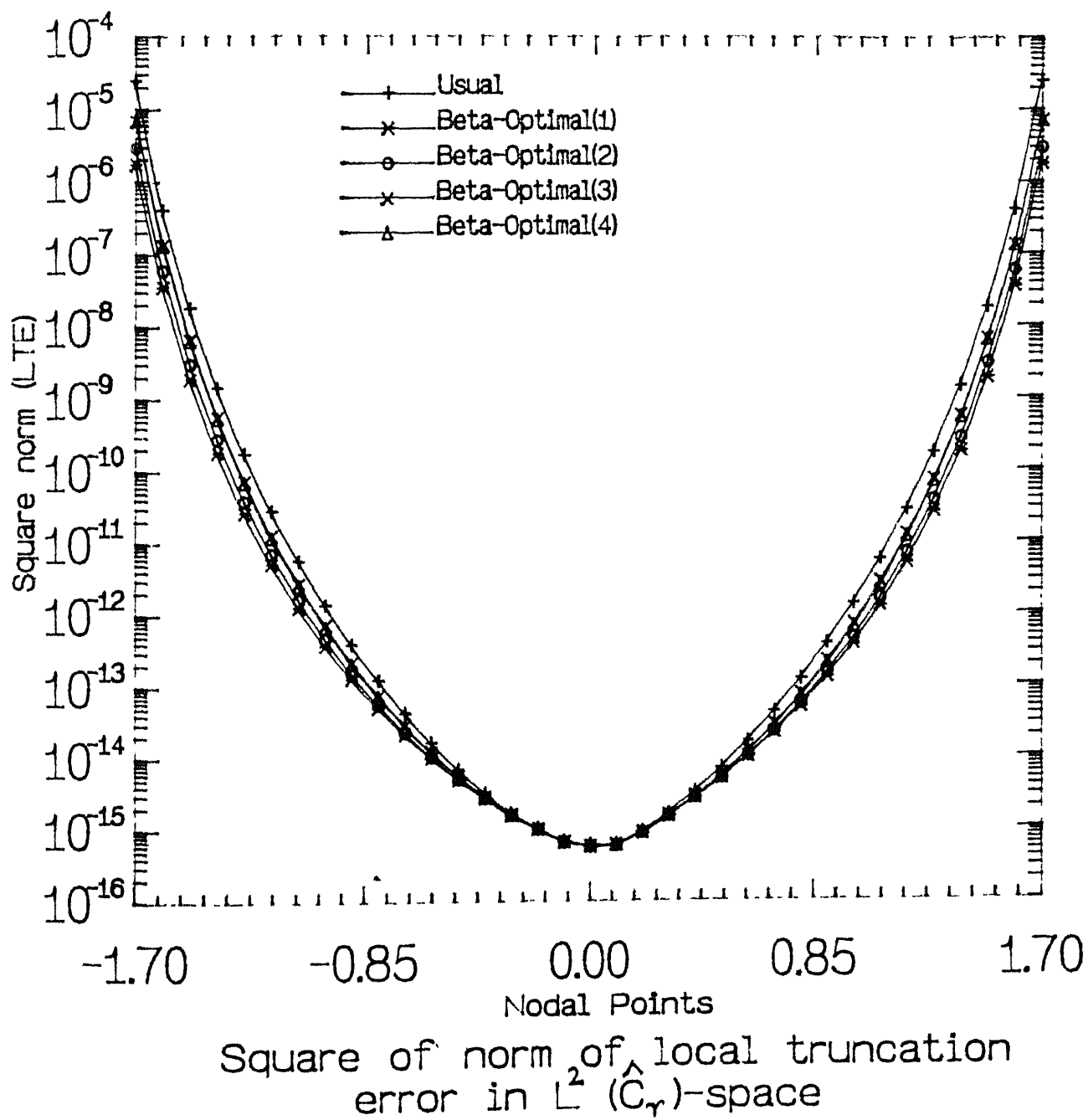


FIG. 1

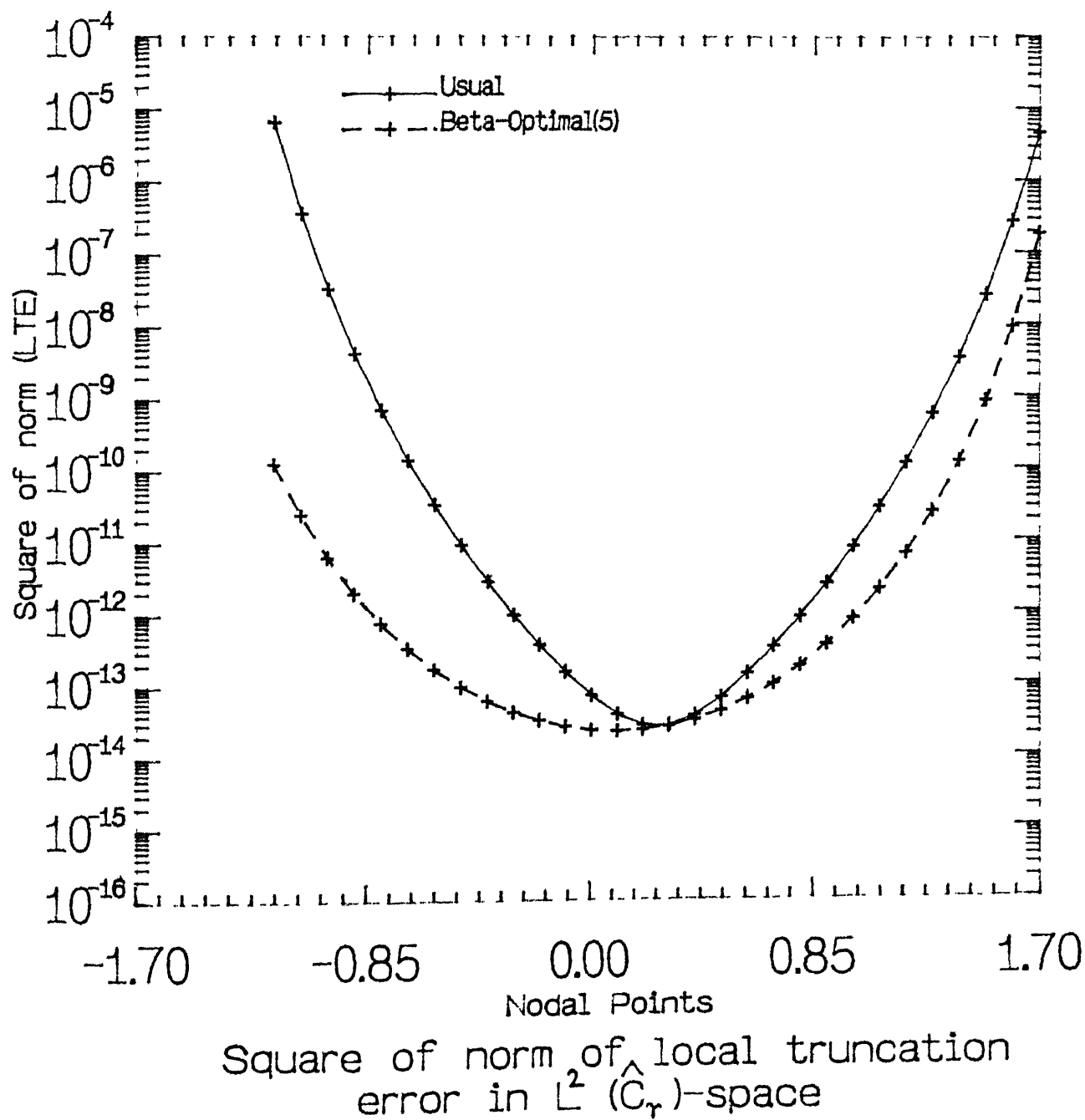


FIG. 2

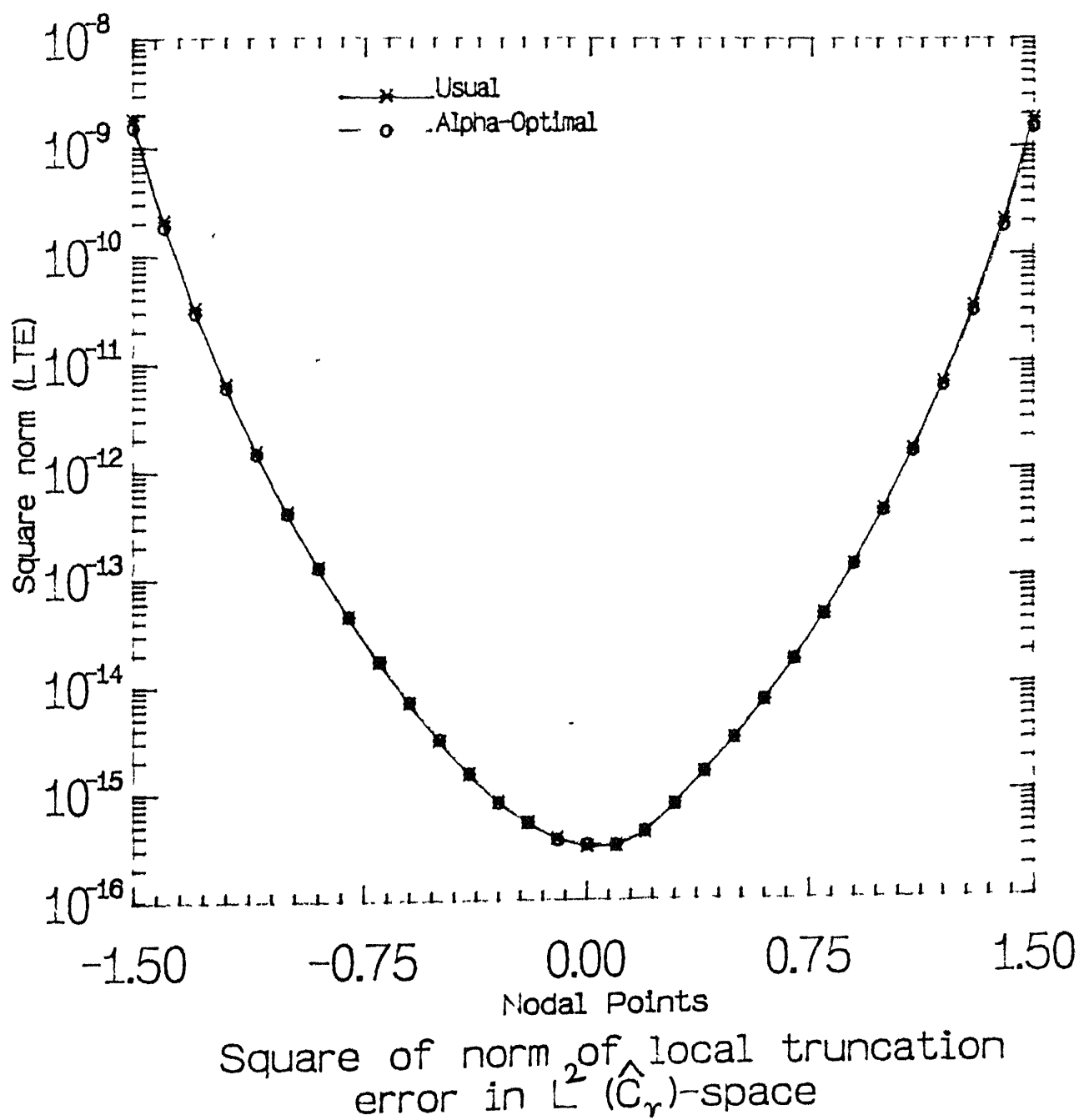


FIG. 3

To obtain the coefficients  $\hat{\alpha}_{j,n}^F$ ,  $j = 0(1)k$  of an  $\alpha$ -optimal multistep method

$$(15) \quad \sum_{j=0}^k \hat{\alpha}_{j,n}^F Y_{n+k-j} + h^2 \sum_{j=\delta_{t0}}^k \beta_j f_{n+k-j} = 0,$$

in  $L^2(\hat{C}_r)$  space with prefixed  $\beta_j$ 's, interpolatory for  $q$  number of linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$ , we have to solve a system of linear equations given in the matrix form as

$$\begin{bmatrix} A & F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} a \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where  $A = (A_{ij})_{\substack{i=0(1)k \\ j=0(1)k}}$ , with  $A_{ij} = \frac{r^2}{\pi(r^2 - x_{n+k-j} \bar{x}_{n+k-1})}$ ;

$F = (F_{ij})_{\substack{i=1(1)q \\ j=0(1)k}}$ , with  $F_{ij} = f_i(x_{n+k-j})$ ;

$a = (\hat{\alpha}_{0,n}^F, \hat{\alpha}_{1,n}^F, \dots, \hat{\alpha}_{k,n}^F)^T$ ;  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T$ ;

$c = (c_0, c_1, \dots, c_k)^T$ , with  $c_i = -h^2 \sum_{j=\delta_{t0}}^k \beta_j \frac{6r^2}{\pi} \frac{x_{n+k-1}^2}{(r^2 - x_{n+k-1} \bar{x}_{n+k-j})^3}$ ;

and

$f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T$ ; with  $\tilde{f}_i = -h^2 \sum_{j=\delta_{t0}}^k \beta_j f_i''(x_{n+k-j})$ .

This method is characterized as follows.

**Theorem 10:** An  $\alpha$ -optimal multistep method (15), in  $L^2(\hat{C}_r)$  space interpolatory for arbitrary functions  $f_1, f_2, \dots, f_q$  is characterized by that it is locally interpolatory for the functions

$$\{f_1, f_2, \dots, f_q\} \cup \{h_i, i = 0(1)k-q\},$$

where

$$h_1(x) = \frac{1}{(r^2 - x\bar{x}_{n+k-1})^2} - \sum_{j=k-q+1}^k \bar{g}_{j+q-k, 1+1} \frac{1}{(r^2 - x\bar{x}_{n+k-j})^2},$$

and  $G = P^{-1}E = (g_{lj})_{\substack{l=1(1)q \\ j=1(1)k-q-\delta_{t_0}+1}}, \text{ where}$

$$P = \left( f_l(x_{n+k-j}) \right)_{\substack{l=1(1)q \\ j=k-q+1(1)k}}, \quad \text{and} \quad E = \left( f_l(x_{n+k-j}) \right)_{\substack{l=1(1)q \\ j=0(1)k-q}}.$$

Proof: The proof follows from Theorem 9 of Chapter 1.

### 5.6 Behavior of the Coefficients as $r \rightarrow \infty$

In section 4.6, we have studied a limiting behavior, as  $r \rightarrow \infty$ , of the coefficients in the optimal multistep method in the space  $H^2(C_r)$ . Analogs of those results also remain valid for the space  $L^2(\hat{C}_r)$  with the only change in the proofs that instead of the orthonormal basis  $\{\psi_k\}$  for  $H^2(C_r)$  space, here the orthonormal basis  $\{\psi_k\}$  for  $L^2(\hat{C}_r)$  space is required. So, in this section we just give the statements of the corresponding results in  $L^2(\hat{C}_r)$  space and omit their proofs.

Theorem 11: Let

$$(16) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \beta_j^F f_{n+k-j} = 0$$

be a multistep method interpolatory for functions  $\{\phi_{\delta_{t_0}}, \dots, \phi_k\}$

and let the matrix  $[\phi_l''(x_{n+k-j})]_{l,j=\delta_{t_0}(1)k}$  be nonsingular. Let

$$(17) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^F f_{n+k-j} = 0$$

be an optimal multistep formula in  $L^2(\hat{C}_r)$  space, interpolatory for functions  $\{\phi_{\delta_{t_0}}, \dots, \phi_{p-1+\delta_{t_0}}\}$ ,  $0 \leq p \leq k + \delta_{t_1} = k + 1 - \delta_{t_0}$ , where  $p=0$  implies the non-interpolatory case for these functions. If the local truncation error functional  $\hat{T}_{nr}^F$  for (17) satisfies

$$(18) \quad \lim_{r \rightarrow \infty} \hat{T}_{nr}^F(\phi_j) = 0, \quad j = p + \delta_{t_0}(1)k$$

then

$$(19) \quad \lim_{r \rightarrow \infty} \hat{\beta}_{j,r}^F = \beta_j^F, \quad j = \delta_{t_0}(1)k$$

Theorem 12: Let  $\alpha_j$ ,  $j = 0(1)k$  be constants such that  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j\alpha_j = 0$ . Then the  $L^2(\hat{C}_r)$ -coefficients  $\hat{\beta}_{j,r}$  of an optimal multistep method

$$(20) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,r}^F f_{n+k-j} = 0$$

approach the coefficients  $\beta_j$  of the usual multistep method

$$(21) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0$$

of maximal polynomial precision.

Theorem 13: Let  $\alpha_j$ ,  $j = 0(1)k$  be constants such that  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j\alpha_j = 0$ . Then the  $L^2(\hat{C}_r)$ -coefficients  $\hat{\beta}_{j,r}^P$  of an optimal multistep method



$$(22) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{j,r}^p f_{n+k-j} = 0,$$

interpolatory for polynomials of degree  $p \leq k + \delta_{t1} + 1$ , approach, as  $r \rightarrow \infty$ , the coefficients  $\beta_j$  of the corresponding usual multistep method

$$\sum_{j=0}^k \alpha_j Y_{n+k-j} - h^2 \sum_{j=\delta_{t0}}^k \beta_j f_{n+k-j} = 0.$$

Theorem 14: Let  $\alpha_j$ ,  $j = 0(1)k$  be constants such that  $\sum_{j=0}^k \alpha_j = 0$ , and  $\sum_{j=1}^k j\alpha_j = 0$ . Let the  $(k + \delta_{t1}) \times (k + \delta_{t1})$  matrix  $M$  be nonsingular, where

$$M = \begin{bmatrix} \phi_1''(x_{n+k-1+\delta_{t1}}) & \dots & \phi_1''(x_n) \\ \dots & \dots & \dots \\ \phi_p''(x_{n+k-1+\delta_{t1}}) & \dots & \phi_p''(x_n) \\ 1 & \dots & 1 \\ x_{n+k-1+\delta_{t1}} & \dots & x_n \\ \dots & \dots & \dots \\ x_{n+k-1+\delta_{t1}}^{k-p-\delta_{t0}} & \dots & x_n^{k-p-\delta_{t0}} \end{bmatrix}$$

Then the coefficients  $\hat{\beta}_{k-j,r}^F$ 's of the optimal multistep method in  $L^2(\hat{C}_r)$  space

$$(23) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} - h^2 \sum_{j=\delta_{t0}}^k \hat{\beta}_{j,r}^F f_{n+k-j} = 0,$$

interpolatory for functions  $\{\phi_1, \phi_2, \dots, \phi_p\}$ ,  $p \leq k + \delta_{t1}$ , approach the coefficients  $\beta_j^F$ 's of the unique multistep method

$$(24) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} - h^2 \sum_{j=\delta_{t_0}}^k \beta_j^F f_{n+k-j} = 0,$$

which is interpolatory for functions

$$\{\phi_1, \phi_2, \dots, \phi_p, x^2, x^3, \dots, x^{k+\delta_{t_1}+1-p}\}.$$

OPTIMAL MULTISTEP METHODS IN  $H_{a,b}^{(m)}$  SPACE

6.1 Introduction

Optimal quadrature rules in the Hilbert space  $H_{0,1}^{(2)}$ , have been studied by Larkin [126], where  $H_{0,1}^{(2)}$  is the Hilbert space of real valued functions, of a real variable  $x$ , having square integrable second derivatives over the interval  $[0,1]$ , with an inner product defined by:

$$(f(x), g(x)) = \alpha f(0)g(0) + \beta f'(0)g'(0) + \int_0^1 f''(y)g''(y)dy,$$

where  $\alpha$  and  $\beta$  are positive real constants.

In this chapter, we define the Hilbert space  $H_{a,b}^{(m)}$ , to be the class of real valued functions, of a real variable  $x$ , having square integrable  $m$ -th derivatives over the interval  $[a,b]$ , with an inner product defined by

$$(f(x), g(x)) = \sum_{i=0}^{m-1} c_i f^{(i)}(a) g^{(i)}(a) + \int_a^b f^{(m)}(y) g^{(m)}(y) dy,$$

where  $c_i$ ,  $i = 0(1)m-1$  are positive real constants.

In section 6.2, we obtain the reproducing kernel function for the Hilbert space  $H_{a,b}^{(m)}$  and verify that the second derivative of this kernel is indeed the representer of the second derivative evaluation functional.

In this chapter, we discuss various optimal multistep methods, discussed in Chapter 1, in Hilbert spaces  $H_{a,b}^{(m)}$  for  $m \geq 3$ . The methodology for convergence for  $\beta$ -optimal multistep method as given in Chapter 3 (Theorem 5 and 6) are valid in a Hilbert space,  $H$ , in

which  $D_x^{p+2}$ , the  $p+2$ -th derivative evaluation functional at  $x$  exists and is uniformly bounded on  $[a,b]$ , where  $p$  is the order of the linear difference operator associated with the corresponding usual method. In  $H_{a,b}^{(m)}$ -space,  $D_x^{m-1}$ , the representer for the  $(m-1)$ -th derivative evaluation functional at  $x$  exists and is bounded on a closed subinterval of the interval  $[a,b]$ . As we know,  $p$  equals 2 for Stormer's usual method with function evaluation at one point and  $p$  equals 4 for Cowell's usual method with function evaluation at three points, the hypothesis of convergence are not satisfied for the functions in the spaces  $H_{a,b}^{(m)}$  for  $m = 3$  and 4 in case of Stormer's method and in the spaces  $H_{a,b}^{(m)}$  for  $m = 3, 4, 5$  and 6 in case of Cowell's method. But still the numerical results on these spaces suggest that optimal methods perform better than the corresponding usual method.

We discuss  $\beta$ -optimal multistep methods in the spaces  $H_{a,b}^{(m)}$  and their numerical illustrations in section 6.3,  $\beta$ -optimal multistep methods interpolatory for polynomials of certain degree in the spaces  $H_{a,b}^{(m)}$  and their numerical implementations in section 6.4, and  $\beta$ -optimal multistep methods interpolatory for linearly independent functions in the spaces  $H_{a,b}^{(m)}$  and their numerical implementations in section 6.5.

We shall implement numerically various optimal methods viz.,  $\beta$ -optimal method,  $\beta$ -optimal method with restriction,  $\beta$ -optimal method interpolatory for polynomials of certain degree,  $\beta$ -optimal method interpolatory for linearly independent functions, taking Cowell's method with function evaluation at three points and Stormer's method with function evaluation at one point as the usual

method, on the boundary value problems with differential equations of the form,

$$\frac{d^2 y}{dx^2} = f(x, y),$$

whose true solutions are given as follows.

1.  $y = |x - .5|^3,$
2.  $y = |x - .5|^3 + |x + .5|^3,$
3.  $y = |x - 1|^3 + |x + 1|^3,$
4.  $y = |x - 1.5|^3 + |x + 1.5|^3,$
5.  $y = |x - .5|^3 + |x - 1|^3 + |x - 1.5|^3,$
6.  $y = |x - .6|^3,$
7.  $y = |x - 1|^{2.7},$
8.  $y = |x - 1|^{2.8},$
9.  $y = |x - 1|^{2.9},$
10.  $y = |x - 1|^{3.1}.$

With the true solution of the form  $y = |x - a|^b$ , we take the associated differential equation of the form

$$y'' = g(x) = b(b-1)|x - a|^{b-2}.$$

For the numerical implementation of  $\beta$ -optimal methods interpolatory for arbitrary functions and  $\beta$ -optimal methods corresponding to Stormer's one point method, we use different sets of differential equations.

In our numerical experiments corresponding to Cowell's and Stormer's usual methods, the optimal beta coefficients in a higher derivative space, such as  $H_{a,b}^{(m)}$ ,  $m \geq 6$ , turn out to be nearly equal to the corresponding usual coefficients and consequently both the methods produce almost equal numerical results including some round off errors. So, for numerical implementation, we are interested in

a lower derivative space such as  $H_{a,b}^{(m)}$ ,  $m=3,4,5$ . For the numerical experimentation in space  $H_{a,b}^{(m)}$ , we take the parameters  $c_i$ 's as 1. We use the notations in the tables as 'r-sum' for the row sum of  $\hat{\beta}_{i,n}$ ,  $i=1,2,3$ ,  $\|T_n\|^2$  and  $\|\hat{T}_n\|^2$  for the square norm of local truncation error functionals for usual and  $\beta$ -optimal methods respectively,  $\bar{e}_u$  and  $\bar{e}_0$  for the average of absolute discretization errors for usual and  $\beta$ -optimal methods respectively.

## 6.2 Derivation of Kernel and the Representer of the Second Derivative Evaluation Functional.

The reproducing kernel function for the Hilbert space  $H_{a,b}^{(m)}$  ought to satisfy the relation

$$(1) \quad f(x) = (f(y), K(y, x)) = \sum_{i=0}^{m-1} c_i f^{(i)}(a) K^{(i)}(a, x) + \int_a^b f^{(m)}(y) K^{(m)}(y, x) dy.$$

Now, on integration by parts,

$$\begin{aligned} \int_a^b f^{(m)}(y) K^{(m)}(y, x) dy &= \sum_{i=0}^{m-1} (-1)^i K^{(m+1)}(y, x) f^{(m-i-1)}(y) + \\ &(-1)^m \int_a^b K^{(2m)}(y, x) f(y) dy. \end{aligned}$$

Hence,

$$\begin{aligned} \int_a^b f^{(m)}(y) K^{(m)}(y, x) dy &= \sum_{i=0}^{m-1} (-1)^i K^{(m+1)}(y, x) f^{(m-i-1)}(y) \Big|_a^b - \\ &\sum_{i=0}^{m-1} (-1)^i K^{(m+1)}(y, x) f^{(m-i-1)}(y) \Big|_{x^-}^{x^+} + (-1)^m \int_a^b K^{(2m)}(y, x) f(y) dy \\ &= \sum_{i=0}^{m-1} (-1)^i K^{(m+1)}(b, x) f^{(m-i-1)}(b) - \sum_{i=0}^{m-1} (-1)^i K^{(m+1)}(a, x) f^{(m-i-1)}(a) \end{aligned}$$

$$\begin{aligned}
& - \sum_{i=0}^{m-1} (-1)^i f^{(m-i-1)}(x) \left[ K^{(m+1)}(x^+, x) - K^{(m+1)}(x^-, x) \right] \\
& + (-1)^m \int_a^b K^{(2m)}(y, x) f(y) dy.
\end{aligned}$$

So, from (1),

$$\begin{aligned}
f(x) &= \sum_{i=0}^{m-1} c_i f^{(i)}(a) K^{(i)}(a, x) + \sum_{i=0}^{m-1} (-1)^i K^{(m+1)}(b, x) f^{(m-i-1)}(b) \\
& - \sum_{i=0}^{m-1} (-1)^{m-i-1} K^{(2m-i-1)}(a, x) f^{(i)}(a) \\
& - \sum_{i=0}^{m-1} (-1)^{m-i-1} f^{(i)}(x) \left[ K^{(2m-i-1)}(x^+, x) - K^{(2m-i-1)}(x^-, x) \right] \\
& + (-1)^m \int_a^b K^{(2m)}(y, x) f(y) dy \\
&= \sum_{i=0}^{m-1} f^{(i)}(a) \left[ c_i K^{(i)}(a, x) + (-1)^{m-i} K^{(2m-i-1)}(a, x) \right] \\
& + \sum_{i=0}^{m-1} (-1)^i K^{(m+1)}(b, x) f^{(m-i-1)}(b) \\
& - \sum_{i=0}^{m-1} (-1)^{m-i-1} f^{(i)}(x) \left[ K^{(2m-i-1)}(x^+, x) - K^{(2m-i-1)}(x^-, x) \right] \\
& + (-1)^m \int_a^b K^{(2m)}(y, x) f(y) dy.
\end{aligned}$$

Comparing both sides we get the following equations:

$$(2) \quad c_i K^{(i)}(a, x) + (-1)^{m-i} K^{(2m-i-1)}(a, x) = 0, \quad i = 0(1)m-1,$$

$$(3) \quad K^{(m+1)}(b, x) = 0, \quad i = 0(1)m-1,$$

$$(4) \quad K^{(2m-i-1)}(x^+, x) - K^{(2m-i-1)}(x^-, x) = 0, \quad i = 1(1)m-1,$$

$$(5) \quad (-1)^{m-1} \left[ K^{(2m-1)}(x^-, x) - K^{(2m-1)}(x^+, x) \right] = 1,$$

$$(6) \quad K^{(2m)}(y,x) = 0,$$

also by continuity of the kernel  $K(y,x)$  at  $x$ , we have

$$(7) \quad K^{(1)}(x^+,x) = K^{(1)}(x^-,x), \quad i = 0(1)m-1.$$

By (6), we have

$$K(y,x) = \begin{cases} \sum_{j=0}^{2m-1} \tilde{\beta}_j y^j, & a \leq y \leq x \\ \sum_{j=0}^{2m-1} \tilde{\gamma}_j y^j, & x \leq y \leq b \end{cases}$$

Setting  $y = a + (b-a)t$  and  $x = a + (b-a)s$ , where  $0 \leq s,t \leq 1$ , we can write

$$K(y,x) = K_1(t,s) = \begin{cases} \sum_{j=0}^{2m-1} \beta_j t^j, & 0 \leq t \leq s \\ \sum_{j=0}^{2m-1} \gamma_j t^j, & s \leq t \leq 1 \end{cases}$$

or,

$$(8) \quad K(y,x) = K_1(t,s) = \begin{cases} \sum_{j=0}^{2m-1} \beta_j \left( \frac{y-a}{b-a} \right)^j, & 0 \leq \frac{y-a}{b-a} \leq \frac{x-a}{b-a} \\ \sum_{j=0}^{2m-1} \gamma_j \left( \frac{y-a}{b-a} \right)^j, & \frac{x-a}{b-a} \leq \frac{y-a}{b-a} \leq 1 \end{cases}$$

$$K^{(1)}(y,x) = \frac{1}{(b-a)^1} K_1^{(1)}(t,s)$$

By (2), we have, for  $i = 0(1)m-1$

$$c_1 \frac{1}{(b-a)^1} \beta_1 i! + (-1)^{m-1} \frac{1}{(b-a)^{2m-1-1}} \beta_{2m-1-1} (2m-i-1)! = 0.$$

$$\text{So, } \beta_{2m-1-1} = (-1)^{m-1-1} (b-a)^{2m-2i-1} \frac{c_1 i!}{(2m-i-1)!} \beta_1, \quad i = 0(1)m-1.$$



By (3), we have

$$\gamma_{m+1} \frac{(m+i)!}{(b-a)^{m+1}} + \frac{\gamma_{m+1+1}}{(b-a)^{m+1}} \frac{(m+i+1)!}{1!} + \frac{\gamma_{m+1+2}}{(b-a)^{m+1}} \frac{(m+i+2)!}{2!} + \dots$$

$$+ \frac{\gamma_{2m-1}}{(b-a)^{m+1}} \frac{(2m-1)!}{(m-i-1)!} = 0, \quad i = 0(1)m-1.$$

So,  $\gamma_m = \gamma_{m+1} = \dots = \gamma_{2m-1} = 0$ .

By (8),  $K(y, x)$  can be written as

$$(9) \quad K(y, x) = \begin{cases} \sum_{j=0}^{m-1} \beta_j \left[ \left( \frac{y-a}{b-a} \right)^j + \frac{(-1)^{m-j-1} (b-a)^{2m-2j-1} c_j j!}{(2m-j-1)!} \left( \frac{y-a}{b-a} \right)^{2m-j-1} \right], & \text{for } 0 \leq \frac{y-a}{b-a} \leq \frac{x-a}{b-a} \\ \sum_{j=0}^{m-1} \gamma_j \left( \frac{y-a}{b-a} \right)^j, & \text{for } \frac{x-a}{b-a} \leq \frac{y-a}{b-a} \leq 1. \end{cases}$$

By (5), we have

$$(-1)^{m-1} (-1)^{m-1} \beta_0 \frac{(b-a)^{2m-1} c_0}{(2m-1)!} \frac{(2m-1)!}{(b-a)^{2m-1}} = 1.$$

So,

$$(10) \quad \beta_0 = \frac{1}{c_0}.$$

Since  $K^{(2m-1-1)}(x^+, x) = 0$ , for  $i = 1(1)m-1$ , we have, from (4)

$$(11) \quad K^{(2m-2)}(x^-, x) = K^{(2m-3)}(x^-, x) = \dots = K^{(m)}(x^-, x).$$

Now,  $K^{(2m-2)}(x^-, x) = 0$ , gives

$$\beta_0 \frac{(-1)^{m-1} (b-a)^{2m-1} c_0}{(2m-1)!} \frac{(2m-1)!}{1! (b-a)^{2m-2}} \frac{x-a}{b-a} +$$

$$\beta_1 \frac{(-1)^{m-2} (b-a)^{2m-3} c_1 1!}{(2m-2)!} \frac{(2m-2)!}{(b-a)^{2m-2}} = 0.$$

Using (10), we get

$$(12) \quad \beta_1 = (b-a)^2 \frac{1}{c_1} \frac{x-a}{b-a}.$$

Further  $K^{(2m-3)}(x^-, x) = 0$ , gives

$$\begin{aligned} \beta_0 \frac{(-1)^{m-1} (b-a)^{2m-1} c_0 0!}{(2m-1)!} \frac{(2m-1)!}{2! (b-a)^{2m-3}} \left( \frac{x-a}{b-a} \right)^2 + \\ + \beta_1 \frac{(-1)^{m-2} (b-a)^{2m-3} c_1 1!}{(2m-2)!} \frac{(2m-2)!}{(b-a)^{2m-3}} \frac{x-a}{b-a} + \\ + \beta_2 \frac{(-1)^{m-3} (b-a)^{2m-5} c_2 2!}{(2m-3)!} \frac{(2m-3)!}{(b-a)^{2m-3}} = 0. \end{aligned}$$

Using (10) and (12), we get

$$(13) \quad \beta_2 = \frac{(b-a)^4}{2!2!c_2} \left( \frac{x-a}{b-a} \right)^2.$$

Let us assume that

$$(14) \quad \beta_i = \frac{1}{i!i!} \frac{(b-a)^{2i}}{c_i} \left( \frac{x-a}{b-a} \right)^i; \quad \text{for all } i = 0(1)j.$$

Now  $K^{(2m-j-2)}(x^-, x) = 0$ , from (11), gives:

$$\begin{aligned} \beta_{j+1} c_{j+1} (-1)^{m-j-2} \frac{(b-a)^{2m-2j-3}}{(2m-j-2)!} \frac{(j+1)!(2m-j-2)!}{0!} + \\ \beta_j c_j (-1)^{m-j-1} \frac{(b-a)^{2m-2j-1}}{(2m-j-1)!} \frac{j!(2m-j-1)!}{1!} \left( \frac{x-a}{b-a} \right) + \\ \beta_{j-1} c_{j-1} (-1)^{m-j} \frac{(b-a)^{2m-2j+1}}{(2m-j)!} \frac{(j-1)!(2m-j)!}{2!} \left( \frac{x-a}{b-a} \right)^2 + \dots + \\ \beta_1 c_1 (-1)^{m-2} \frac{(b-a)^{2m-3}}{(2m-2)!} \frac{1!(2m-2)!}{j!} \left( \frac{x-a}{b-a} \right)^j + \\ \beta_0 c_0 (-1)^{m-1} \frac{(b-a)^{2m-1}}{(2m-1)!} \frac{0!(2m-1)!}{(j+1)!} \left( \frac{x-a}{b-a} \right)^{j+1} = 0. \end{aligned}$$

Substituting the values of  $\beta_i$ ,  $i = 0(1)j$ , we get

$$\beta_{j+1} c_{j+1} (-1)^{m-j-2} (b-a)^{2m-2j-3} (j+1)! + (-1)^{m-j-1} \frac{(b-a)^{2m-1}}{1!j!} \left( \frac{x-a}{b-a} \right)^{j+1} +$$

$$\begin{aligned}
& (-1)^{m-j} \frac{(b-a)^{2m-1}}{2!(j-1)!} \left( \frac{x-a}{b-a} \right)^{j+1} + (-1)^{m-j+1} \frac{(b-a)^{2m-1}}{3!(j-2)!} \left( \frac{x-a}{b-a} \right)^{j+1} + \dots \\
& + (-1)^{m-2} \frac{(b-a)^{2m-1}}{j!1!} \left( \frac{x-a}{b-a} \right)^{j+1} + (-1)^{m-1} \frac{(b-a)^{2m-1}}{(j+1)!} \left( \frac{x-a}{b-a} \right)^{j+1} = 0.
\end{aligned}$$

Cancelling  $(-1)^m (b-a)^{2m-1}$  throughout we get

$$\begin{aligned}
(-1)^{-j-2} \beta_{j+1} c_{j+1} &= \frac{1}{(j+1)!} (b-a)^{2(j+1)} \left( \frac{x-a}{b-a} \right)^{j+1} \\
&\cdot \left[ \frac{1}{(j+1)!} - \frac{1}{j!1!} + \frac{1}{(j-1)!2!} - \dots + \frac{(-1)^{-j+2}}{3!(j-2)!} + \frac{(-1)^{-j+1}}{2!(j-1)!} + \frac{(-1)^{-j}}{1!j!} \right] \\
&= \frac{1}{(j+1)!} (b-a)^{2(j+1)} \left( \frac{x-a}{b-a} \right)^{j+1} \\
&\cdot \left[ \frac{1}{(j+1)!} - \frac{1}{j!1!} + \frac{1}{(j-1)!2!} - \dots + \frac{(-1)^{j-2}}{3!(j-2)!} + \frac{(-1)^{j-1}}{2!(j-1)!} + \frac{(-1)^j}{1!j!} \right].
\end{aligned}$$

We know,

$$0 = (1-1)^{j+1} = 1 - {}^{j+1}C_1 + {}^{j+1}C_2 - \dots + (-1)^j {}^{j+1}C_j + (-1)^{j+1} {}^{j+1}C_{j+1}.$$

So,

$$\frac{1}{(j+1)!} - \frac{1}{j!1!} + \frac{1}{(j-1)!2!} - \dots + \frac{(-1)^{j-1}}{2!(j-1)!} + \frac{(-1)^j}{1!j!} = \frac{(-1)^{j+2}}{(j+1)!}.$$

Hence,

$$(15) \quad \beta_{j+1} c_{j+1} = \frac{1}{[(j+1)!]^2} (b-a)^{2(j+1)} \left( \frac{x-a}{b-a} \right)^{j+1}.$$

By (10), (12), (13), (14) and (15) the induction follows. Thus

$$(16) \quad \beta_i = \frac{1}{i!i!} \frac{(b-a)^{2i}}{c_i} \left( \frac{x-a}{b-a} \right)^i; \quad \text{for all } i = 0(1)m-1$$

Substituting the values of  $\beta_i$ ,  $i = 0(1)m-1$ , in (9), and writing  $t = \frac{y-a}{b-a}$  and  $s = \frac{x-a}{b-a}$ , we get

$$K(y, x) = \begin{cases} \frac{1}{c_0} \left[ 1 + \frac{(-1)^{m-1} (b-a)^{2m-1} c_0}{(2m-1)!} t^{2m-1} \right] + \\ \sum_{j=1}^{m-1} \frac{s^j (b-a)^{2j}}{j! j! c_j} \left[ t^j + \frac{(-1)^{m-j-1} (b-a)^{2m-2j-1} c_j j!}{(2m-j-1)!} t^{2m-j-1} \right], & \text{for } 0 \leq t \leq s, \\ \sum_{j=0}^{m-1} r_j t^j, & \text{for } s \leq t \leq 1. \end{cases}$$

As  $f(x) = (f(t), K(t, x))$ , we have

$$(K(t, s), K(t, x)) = K(x, s).$$

Again in a real Hilbert space, inner product being symmetric, we have

$$(K(t, s), K(t, x)) = (K(t, x), K(t, s)) = K(s, x).$$

Thus,  $K(x, s) = K(s, x)$ . Hence, we can write

$$(17) \quad K(y, x) = \begin{cases} \frac{1}{c_0} \left[ 1 + \frac{(-1)^{m-1} (b-a)^{2m-1} c_0}{(2m-1)!} t^{2m-1} \right] + \\ \sum_{j=1}^{m-1} \frac{s^j (b-a)^{2j}}{j! j! c_j} \left[ t^j + \frac{(-1)^{m-j-1} (b-a)^{2m-2j-1} c_j j!}{(2m-j-1)!} t^{2m-j-1} \right], & \text{for } 0 \leq t \leq s \\ \frac{1}{c_0} \left[ 1 + \frac{(-1)^{m-1} (b-a)^{2m-1} c_0}{(2m-1)!} s^{2m-1} \right] + \\ \sum_{j=1}^{m-1} \frac{t^j (b-a)^{2j}}{j! j! c_j} \left[ s^j + \frac{(-1)^{m-j-1} (b-a)^{2m-2j-1} c_j j!}{(2m-j-1)!} s^{2m-j-1} \right], & \text{for } s \leq t \leq 1 \end{cases}$$

$$D^2(Y, x) = \frac{\partial^2}{\partial x^2} K(Y, x) \Big|_x.$$

$$(18) \quad D_2(Y, X) = \begin{cases} \sum_{j=2}^{m-1} \frac{(-1)^{m-j-1} (b-a)^{2m-3} s^{j-2} t^{2m-j-1}}{(j-2)! (2m-j-1)!} + \sum_{j=2}^{m-1} \frac{(b-a)^{2j-2} s^{j-2} t^j}{(j-2)! j! c_j} & \text{for } 0 \leq t \leq s \\ \frac{(-1)^{m-1} (b-a)^{2m-3} s^{2m-3}}{(2m-3)!} + \sum_{j=1}^{m-1} \frac{(-1)^{m-j-1} (b-a)^{2m-3} s^{2m-j-3} t^j}{j! (2m-j-3)!} + \\ \sum_{j=2}^{m-1} \frac{(b-a)^{2j-2} s^{j-2} t^j}{(j-2)! j! c_j}, & \text{for } s \leq t \leq 1 \end{cases}$$

For  $i = 1(1)m-1$ , we have

$$D_2^{(1)}(Y, X) = \frac{\partial^2}{\partial Y^2} D_2(Y, X) \Big|_X.$$

$$(19) \quad D_2^{(1)}(Y, X) = \begin{cases} \sum_{j=2}^{m-1} \frac{(-1)^{m-j-1} (b-a)^{2m-3-1} s^{j-2} t^{2m-j-1-1}}{(j-2)! (2m-j-1-1)!} + \\ \sum_{j=1}^{m-1} \frac{(b-a)^{2j-2-1} s^{j-2} t^{j-1}}{(j-2)! (j-1)! c_j}, & \text{for } 0 \leq t \leq s, \\ \sum_{j=1}^{m-1} \frac{(-1)^{m-j-1} (b-a)^{2m-3-1} s^{2m-j-3} t^{j-1}}{(j-1)! (2m-j-3)!} + \\ \sum_{j=1}^{m-1} \frac{(b-a)^{2j-2-1} s^{j-2} t^{j-1}}{(j-2)! (j-1)! c_j}, & \text{for } s \leq t \leq 1. \end{cases}$$

Now for the space  $H_{a,b}^{(m)}$ , we directly verify, in the following theorem, that  $D_2(Y, X)$ , given by the second derivative of the kernel  $K(Y, X)$  is indeed the representer of the second derivative evaluation functional.

**Theorem 1:** The function  $D_2(Y, X)$  given by (18) is the representer for the second derivative evaluation functional in  $H_{2,c}^{(m)}$  space.

**Proof:** The  $m$ -th derivative of  $D_2(Y, X)$  is given by

$$D2^{(m)}(y, x) = \frac{1}{(b-a)^{m+2}} \begin{cases} \sum_{j=2}^{m-1} \frac{(-1)^{m-j-1} (b-a)^{2m-1} s^{j-2} t^{m-j-1}}{(j-2)!(m-j-1)!}, & \text{for } 0 \leq t \leq s, \\ 0, & \text{for } s \leq t \leq 1. \end{cases}$$

$D2^{(m)}(y, x)$  is square integrable. So,  $D2(y, x) \in H_{a,b}^{(m)}$ . To complete the proof, we need to show that  $f''(z_0) = (f(t), D2(t, z_0))$ , for any point  $z_0 \in [a, b]$ . By (19), we have

$$D2(a, z_0) = 0; \quad D2'(a, z_0) = 0,$$

$$D2^{(i)}(a, z_0) = \frac{(b-a)^{i-2} s^{i-2}}{(i-2)! c_i}, \quad i=2(1)m-1$$

By the definition of the inner product of the space  $H_{2,c}^{(m)}$ , we have,

$$\begin{aligned} & (f(t), D2(t, z_0)) \\ &= \sum_{i=0}^{m-1} c_i f^{(i)}(a) D2^{(i)}(a, z_0) + \int_a^b f^{(m)}(x) D2^{(m)}(x, z_0) dx \\ &= \sum_{i=2}^{m-1} c_i f^{(i)}(a) \frac{(z_0-a)^{i-2}}{(i-2)! c_i} - \int_a^{z_0} \sum_{j=2}^{m-1} (-1)^{m-j-1} \frac{(z_0-a)^{j-2} (x-a)^{m-j-1}}{(j-2)!(m-j-1)!} f^{(m)}(x) dx. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} & (f(t), D2(t, z_0)) = \\ & (z_0-a)^{m-3} f^{(m-1)}(z_0) \left\{ \frac{1}{(m-3)!} - \frac{1}{(m-4)!1!} + \frac{1}{(m-5)!2!} - \dots + \frac{(-1)^{m-3}}{(m-3)!} \right\} \\ & + (z_0-a)^{m-4} f^{(m-2)}(z_0) \left\{ \frac{1}{(m-4)!} - \frac{1}{(m-5)!1!} + \frac{1}{(m-6)!2!} - \dots + \frac{(-1)^{m-4}}{(m-4)!} \right\} \\ & + \dots + (z_0-a) f^{(3)}(z_0) \{1-1\} + f''(z_0). \end{aligned}$$

Using the fact that

$$\frac{1}{j!} - \frac{1}{(j-1)!1!} + \frac{1}{(j-2)!2!} - \dots + \frac{(-1)^{j-1}}{1!(j-1)!} + \frac{(-1)^j}{j!} = 0,$$

we get  $(f(t), D2(t, z_0)) = f''(z_0)$ . Hence the proof.

### 6.3 Optimal Multistep Methods in $H_{a,b}^{(m)}$ Space

In  $H_{a,b}^{(m)}$  space, to determine the optimal coefficients  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k$  of  $\beta$ -optimal multistep method

$$(20) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} f_{n+k-j} = 0,$$

where  $\alpha_j$ 's are prefixed according to some consistent and stable known usual method with highest degree polynomial precision,

$$(21) \quad \sum_{j=0}^k \alpha_j Y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \beta_j f_{n+k-j} = 0.$$

With reference to equation (11) of Chapter 1, we have the system of normal equations given by

$$\hat{c} \hat{b} = \hat{d},$$

where

$$\hat{b} = h^2 \left( \hat{\beta}_{\delta_{t_0},n}, \dots, \hat{\beta}_{k,n} \right)^T,$$

$$\hat{c}_{ij}, \text{ for } i, j = \delta_{t_0}(1)k, \text{ and } \hat{d}_i = - \sum_{j=0}^k \alpha_j \hat{d}_{ij}, \text{ for } i = \delta_{t_0}(1)k \text{ are}$$

defined as follows.

$$\hat{c}_{1j} = \left\{ \begin{array}{l} \sum_{j=2}^{m-1} \frac{(-1)^{m-j-1} (x_{n+k-j} - a)^{2m-j-3} (x_{n+k-1} - a)^{j-2}}{(j-2)!(2m-j-3)!} + \\ \sum_{j=2}^{m-1} \frac{(x_{n+k-1} - a)^{j-2} (x_{n+k-j} - a)^{j-2}}{(j-2)!(j-2)!c_j}; \text{ for } x_{n+k-j} \leq x_{n+k-1} \\ \sum_{j=2}^{m-1} \frac{(-1)^{m-j-1} (x_{n+k-1} - a)^{2m-j-3} (x_{n+k-j} - a)^{j-2}}{(j-2)!(2m-j-3)!} + \\ \sum_{j=2}^{m-1} \frac{(x_{n+k-1} - a)^{j-2} (x_{n+k-j} - a)^{j-2}}{(j-2)!(j-2)!c_j}; \text{ for } x_{n+k-j} \geq x_{n+k-1}. \end{array} \right.$$

and

$$\hat{d}_{1j} = \left\{ \begin{array}{l} \sum_{j=2}^{m-1} \frac{(-1)^{m-j-1} (x_{n+k-j} - a)^{2m-j-1} (x_{n+k-1} - a)^{j-2}}{(j-2)!(2m-j-1)!} + \\ \sum_{j=2}^{m-1} \frac{(x_{n+k-1} - a)^{j-2} (x_{n+k-j} - a)^j}{(j-2)!j!c_j}; \text{ for } x_{n+k-j} \leq x_{n+k-1} \\ \frac{(-1)^{m-1} (x_{n+k-1} - a)^{2m-3}}{(2m-3)!} + \sum_{j=1}^{m-1} \frac{(-1)^{m-j-1} (x_{n+k-1} - a)^{2m-j-3} (x_{n+k-j} - a)^j}{j!(2m-j-3)!} + \\ \sum_{j=2}^{m-1} \frac{(x_{n+k-1} - a)^{j-2} (x_{n+k-j} - a)^j}{(j-2)!j!c_j}; \text{ for } x_{n+k-j} \geq x_{n+k-1}. \end{array} \right.$$

This system of equations may be solved for  $h^2 \hat{\beta}_{j,n}$ ,  $j = \delta_{t_0}(1)k$ .

The following theorem characterizes the  $\beta$ -optimal multistep method (20) in  $H_{a,b}^{(m)}$  space.

Theorem 2: The optimal multistep method (20) in which  $\alpha_j$ 's are prefixed, and optimization is done with respect to  $\beta_j$ 's, in  $H_{a,b}^{(m)}$  space, is characterized by that it is locally interpolatory for



functions  $y_i(x)$ ,  $i = \delta_{t_0}(1)k$  where

$$y_i(x) = \begin{cases} \sum_{j=2}^{m-1} \frac{(-1)^{m-j-1} x^{2m-j-1} (x_{n+k-1} - a)^{j-2}}{(j-2)!(2m-j-1)!} + \sum_{j=2}^{m-1} \frac{(x_{n+k-1} - a)^{j-2} x^j}{(j-2)!j!c}, & \text{for } x \leq x_{n+k-1} \\ \frac{(-1)^{m-1} (x_{n+k-1} - a)^{2m-3}}{(2m-3)!} + \sum_{j=1}^{m-1} \frac{(-1)^{m-j-1} (x_{n+k-1} - a)^{2m-j-3} x^j}{j!(2m-j-3)!} + \\ \sum_{j=2}^{m-1} \frac{(x_{n+k-1} - a)^{j-2} x^j}{(j-2)!j!c_j} & \text{for } x \geq x_{n+k-1}. \end{cases}$$

Proof: The proof follows from Theorem 1 of Chapter 1, using the definitions of  $K(z, \bar{t})$ ,  $D_2(t, \bar{z})$ ,  $D_2''(t, \bar{z})$  from (17), (18), (19).

The numerical results suggest that in space  $H_{a,b}^{(m)}$ , optimal beta coefficients  $\hat{\beta}_{0n}$ ,  $\hat{\beta}_{1n}$ ,  $\hat{\beta}_{2n}$  for  $\beta$ -optimal method and the square norm of local truncation errors functionals for Cowell's usual and the corresponding  $\beta$ -optimal methods are independent of nodal points and the results for  $h=.1$ . are given as follows

$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
.1667E-02	.6667E-02	.1667E-02	.1000E-01	.5833E-06	.4444E-06

It seems that,  $\hat{\beta}_{0n}=1/6$ ,  $\hat{\beta}_{1n}=2/3$ ,  $\hat{\beta}_{2n}=1/6$ .

In the following table we present the numerical results for Cowell's usual and the corresponding  $\beta$ -optimal methods on the equations listed above on interval  $[-2, 2]$  for  $h = .1$ .

Table - 1(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.107441E-13
2	.961538E-02	.290665E-13
3	.769231E-02	.487416E-13
4	.448718E-02	.848324E-13
5	.108974E-01	.497921E-13
6	.466667E-02	.796221E-14
7	.102985E-01	.841119E-02
8	.749346E-02	.507054E-02
9	.539873E-02	.232991E-02
10	.270475E-02	.207524E-02

Table for average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method in  $H_{a,b}^{(3)}$ -space, for 10 BVP-s, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

The above numerical results show that as compared to usual method, the  $\beta$ -optimal method in  $H_{a,b}^{(3)}$ -space, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ , is far superior for BVP's with equations 1-6, one decimal place better for BVP's with equation 7 and just better on equations 8,9,10. If the function of the differential equation considered is not very smooth, then the optimal method is better suggested and if the function is smooth enough, then the usual method is better suggested. If the function is very bad, i.e., too far from smoothness then neither usual nor optimal method will perform better. This situation hold for  $\beta$ -optimization in  $H_{a,b}^{(4)}$ ,  $H_{a,b}^{(5)}$ -space which are seen from the following tables.

Table - 2

	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.9	.1010E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-1.8	.1009E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.7	.1009E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.6	.1009E-02	.7990E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.5	.1009E-02	.7990E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.4	.1009E-02	.7990E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.3	.1009E-02	.7990E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.2	.1009E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.1	.1010E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.0	.1010E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-.9	.1010E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-.8	.1011E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-.7	.1011E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-.6	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-.5	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-.4	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-.3	.1012E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-.2	.1012E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1262E-09
-.1	.1012E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1262E-09
.0	.1012E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1262E-09
.1	.1012E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1262E-09
.2	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1262E-09
.3	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1262E-09
.4	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1262E-09
.5	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
.6	.1011E-02	.7987E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
.7	.1011E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
.8	.1011E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
.9	.1010E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
1.0	.1010E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
1.1	.1010E-02	.7988E-02	.1002E-02	.1000E-01	.1454E-09	.1262E-09
1.2	.1010E-02	.7988E-02	.1002E-02	.1000E-01	.1454E-09	.1262E-09
1.3	.1010E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
1.4	.1010E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
1.5	.1009E-02	.7989E-02	.1002E-02	.1000E-01	.1456E-09	.1264E-09
1.6	.1009E-02	.7989E-02	.1002E-02	.1000E-01	.1456E-09	.1265E-09
1.7	.1009E-02	.7989E-02	.1002E-02	.1000E-01	.1457E-09	.1266E-09
1.8	.1009E-02	.7990E-02	.1002E-02	.1000E-01	.1457E-09	.1266E-09
1.9	.1009E-02	.7990E-02	.1002E-02	.1000E-01	.1458E-09	.1267E-09

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals for Cowell's 3-points usual and the corresponding  $\beta$ -optimal method in  $H_{a,b}^{(4)}$  space, at the nodal points, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.0833333$ ,  $\beta_1=.833333$ ,  $\beta_2=.0833333$ .

Table - 3

$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$	
-1.9	.8363E-03	.8327E-02	.8363E-03	.1000E-01	.8830E-13	.8736E-13
-1.8	.8361E-03	.8328E-02	.8361E-03	.1000E-01	.8795E-13	.8708E-13
-1.7	.8360E-03	.8328E-02	.8360E-03	.1000E-01	.8759E-13	.8676E-13
-1.6	.8360E-03	.8328E-02	.8359E-03	.1000E-01	.8816E-13	.8734E-13
-1.5	.8360E-03	.8328E-02	.8359E-03	.1000E-01	.8860E-13	.8779E-13
-1.4	.8361E-03	.8328E-02	.8360E-03	.1000E-01	.8861E-13	.8777E-13
-1.3	.8363E-03	.8328E-02	.8361E-03	.1000E-01	.8776E-13	.8690E-13
-1.2	.8364E-03	.8327E-02	.8362E-03	.1000E-01	.8804E-13	.8714E-13
-1.1	.8366E-03	.8327E-02	.8364E-03	.1000E-01	.8808E-13	.8713E-13
-1.0	.8368E-03	.8327E-02	.8366E-03	.1000E-01	.8705E-13	.8604E-13
-.9	.8371E-03	.8326E-02	.8368E-03	.1000E-01	.8773E-13	.8666E-13
-.8	.8373E-03	.8326E-02	.8370E-03	.1000E-01	.9176E-13	.9061E-13
-.7	.8376E-03	.8325E-02	.8372E-03	.1000E-01	.9188E-13	.9067E-13
-.6	.8378E-03	.8325E-02	.8374E-03	.1000E-01	.8998E-13	.8870E-13
-.5	.8380E-03	.8324E-02	.8376E-03	.1000E-01	.9181E-13	.9047E-13
-.4	.8382E-03	.8324E-02	.8378E-03	.1000E-01	.8925E-13	.8785E-13
-.3	.8384E-03	.8324E-02	.8379E-03	.1000E-01	.8664E-13	.8520E-13
-.2	.8385E-03	.8323E-02	.8381E-03	.1000E-01	.9066E-13	.8916E-13
-.1	.8387E-03	.8323E-02	.8382E-03	.1000E-01	.9529E-13	.9377E-13
.0	.8388E-03	.8323E-02	.8383E-03	.1000E-01	.9013E-13	.8858E-13
.1	.8388E-03	.8323E-02	.8384E-03	.1000E-01	.8803E-13	.8644E-13
.2	.8389E-03	.8323E-02	.8384E-03	.1000E-01	.9601E-13	.9437E-13
.3	.8389E-03	.8323E-02	.8385E-03	.1000E-01	.7816E-13	.7658E-13
.4	.8389E-03	.8323E-02	.8385E-03	.1000E-01	.8363E-13	.8196E-13
.5	.8390E-03	.8323E-02	.8385E-03	.1000E-01	.9096E-13	.8935E-13
.6	.8389E-03	.8323E-02	.8385E-03	.1000E-01	.1108E-12	.1093E-12
.7	.8389E-03	.8323E-02	.8385E-03	.1000E-01	.1309E-12	.1292E-12
.8	.8389E-03	.8323E-02	.8385E-03	.1000E-01	.6498E-13	.6334E-13
.9	.8389E-03	.8323E-02	.8384E-03	.1000E-01	.7668E-13	.7494E-13
1.0	.8388E-03	.8323E-02	.8384E-03	.1000E-01	.7596E-13	.7455E-13
1.1	.8388E-03	.8323E-02	.8384E-03	.1000E-01	.6712E-13	.6561E-13
1.2	.8388E-03	.8323E-02	.8384E-03	.1000E-01	.1469E-12	.1453E-12
1.3	.8387E-03	.8323E-02	.8383E-03	.1000E-01	.1283E-12	.1269E-12
1.4	.8387E-03	.8323E-02	.8383E-03	.1000E-01	.9255E-13	.9104E-13
1.5	.8386E-03	.8323E-02	.8382E-03	.1000E-01	.1193E-12	.1177E-12
1.6	.8386E-03	.8323E-02	.8382E-03	.1000E-01	.2372E-12	.2354E-12
1.7	.8385E-03	.8323E-02	.8381E-03	.1000E-01	.7615E-13	.7438E-13
1.8	.8384E-03	.8323E-02	.8381E-03	.1000E-01	.6879E-13	.6661E-13
1.9	.8384E-03	.8324E-02	.8380E-03	.1000E-01	.3280E-12	.3264E-12

Table for optimal  $\beta$ 's, their row-sum, and the square norm of the local truncation error functional for Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $H_{a,b}^{(s)}$  space, at nodal points

with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.0833333$ ,

$\beta_1=.833333$ ,  $\beta_2=.0833333$ .

Table - 2(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.325777E-02
2	.961538E-02	.710710E-02
3	.769231E-02	.538832E-02
4	.448718E-02	.254809E-02
5	.108974E-01	.601694E-02
6	.466667E-02	.307797E-02
7	.102985E-01	.937909E-02
8	.749346E-02	.635989E-02
9	.539873E-02	.401227E-02
10	.270475E-02	.857182E-03

Table for average discretisation error using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods in  $H_{a,b}^{(4)}$ -space, for 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

Table - 3(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.475926E-02
2	.961538E-02	.954940E-02
3	.769231E-02	.763332E-02
4	.448718E-02	.443331E-02
5	.108974E-01	.107347E-01
6	.466667E-02	.461616E-02
7	.102985E-01	.102668E-01
8	.749346E-02	.745476E-02
9	.539873E-02	.535184E-02
10	.270475E-02	.263734E-02

Table for average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method in  $H_{a,b}^{(5)}$ -space, for 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

The results for  $\alpha$ -optimal method on the same equations 1-10 are not so promising.

In  $H_{a,b}^{(m)}$  space, to determine the optimal coefficients  $\hat{\beta}_{i,n}$ ,  $i = \delta_{t_0}(1)k$  of  $\beta$ -optimal multistep method (20) subject to the condition

$$(22) \quad \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n} = 1,$$

where  $\alpha_j$ 's are prefixed according to some consistent and stable known usual method (21) with highest degree polynomial precision, with reference to equation (15) of Chapter 1, we have to solve the system of normal equations given by

$$\hat{C} \hat{b} = \hat{d},$$

where

$$\hat{b} = h^2 \left( \hat{\beta}_{\delta_{t_0}, n}, \dots, \hat{\beta}_{k-1, n} \right)^T,$$

$$\hat{C}_{ij} = D2''(x_{n+k-j}, x_{n+k-1}) - D2''(x_{n+k-j}, x_n) - D2''(x_n, x_{n+k-1}) + D2''(x_n, x_n),$$

for  $i, j = \delta_{t_0}(1)k-1,$

$$\hat{d}_i = \sum_{j=0}^k \alpha_j [D2(x_{n+k-j}, x_{n+k-1}) - D2(x_{n+k-j}, x_n)] -$$

$$-h^2 [D2''(x_n, x_{n+k-1}) - D2''(x_n, x_n)], \quad \text{for } i = \delta_{t_0}(1)k-1,$$

and  $D2(y, x)$  and  $D2''(y, x)$  are given by (18) and (19). This system of equations can be solved for  $h^2 \hat{\beta}_{j,n}$ ,  $j = \delta_{t_0}(1)k-1$ .

The following theorem characterizes the  $\beta$ -optimal multistep method (20) in  $H_{a,b}^{(m)}$  space subject to the condition (22).

**Theorem 3:** The optimal multistep method (20) in which  $\alpha_j$ 's are prefixed and  $\beta_j$ 's are optimized in  $H_{a,b}^{(m)}$  space, subject to the condition (22) is characterized by that it is locally interpolatory for functions  $\{D2(x, x_{n+k-i}) - D2(x, x_n), i = \delta_{t_0}(1)k-1\}$ , where  $D2(y, x)$  is given by (18).

**Proof:** The proof follows from Theorem 3 of Chapter 1.

In the following tables we are presenting the numerical

results for  $\beta$ -optimal method with restriction corresponding to Cowell's usual method with function evaluation at three points. The behavior of the performance of  $\beta$ -optimal method with restriction, on the above 10 BVP, over Cowell's usual method is similar to that in the case of  $\beta$ -optimal method, discussed in section 6.3.

In  $H_{a,b}^{(3)}$ -space, the optimal beta coefficients and the square norm of local truncation error functional for usual and  $\beta$ -optimal methods with restriction are independent of points and these quantities are numerically almost the same as in the case of  $\beta$ -optimal method in  $H_{a,b}^{(3)}$ -space, discussed in section 6.3. We therefore omit the corresponding table. The following table shows the performance of usual and  $\beta$ -optimal methods with restriction on the above 10 BVP's.

Table - 4(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.401411E-14
2	.961538E-02	.115378E-13
3	.769231E-02	.219995E-13
4	.448718E-02	.505806E-13
5	.108974E-01	.135070E-13
6	.466667E-02	.685492E-14
7	.102985E-01	.841119E-02
8	.749346E-02	.507054E-02
9	.539873E-02	.232991E-02
10	.270475E-02	.207524E-02

Table for average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal method with restriction, in  $H_{a,b}^{(3)}$ -space, on 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

We are presenting the numerical results, in Tables 5 and 5(i) for  $H_{a,b}^{(4)}$  space and in Tables 6 and 6(i) for  $H_{a,b}^{(5)}$  space for the same 10 BVP's.

Table - 5

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.9	.1010E-02	.7988E-02	.1002E-02	.1000E-01	.1455E-09	.1263E-09
-1.8	.1009E-02	.7989E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.7	.1008E-02	.7990E-02	.1002E-02	.1000E-01	.1455E-09	.1264E-09
-1.6	.1008E-02	.7991E-02	.1002E-02	.1000E-01	.1455E-09	.1265E-09
-1.5	.1007E-02	.7991E-02	.1001E-02	.1000E-01	.1455E-09	.1265E-09
-1.4	.1007E-02	.7992E-02	.1001E-02	.1000E-01	.1455E-09	.1265E-09
-1.3	.1006E-02	.7992E-02	.1001E-02	.1000E-01	.1455E-09	.1266E-09
-1.2	.1006E-02	.7993E-02	.1001E-02	.1000E-01	.1455E-09	.1266E-09
-1.1	.1006E-02	.7993E-02	.1001E-02	.1000E-01	.1455E-09	.1266E-09
-1.0	.1005E-02	.7994E-02	.1001E-02	.1000E-01	.1455E-09	.1266E-09
-.9	.1005E-02	.7994E-02	.1001E-02	.1000E-01	.1455E-09	.1266E-09
-.8	.1005E-02	.7994E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
-.7	.1005E-02	.7994E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
-.6	.1004E-02	.7995E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
-.5	.1004E-02	.7995E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
-.4	.1004E-02	.7995E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
-.3	.1004E-02	.7995E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
-.2	.1004E-02	.7995E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
-.1	.1004E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
.0	.1004E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
.1	.1003E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
.2	.1003E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
.3	.1003E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1267E-09
.4	.1003E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1268E-09
.5	.1003E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1268E-09
.6	.1003E-02	.7996E-02	.1001E-02	.1000E-01	.1455E-09	.1268E-09
.7	.1003E-02	.7997E-02	.1001E-02	.1000E-01	.1455E-09	.1268E-09
.8	.1003E-02	.7997E-02	.1001E-02	.1000E-01	.1455E-09	.1268E-09
.9	.1003E-02	.7997E-02	.1001E-02	.1000E-01	.1455E-09	.1268E-09
1.0	.1003E-02	.7997E-02	.1001E-02	.1000E-01	.1455E-09	.1268E-09
1.1	.1003E-02	.7997E-02	.1001E-02	.1000E-01	.1454E-09	.1267E-09
1.2	.1003E-02	.7997E-02	.1001E-02	.1000E-01	.1454E-09	.1267E-09
1.3	.1002E-02	.7997E-02	.1000E-02	.1000E-01	.1455E-09	.1268E-09
1.4	.1002E-02	.7997E-02	.1000E-02	.1000E-01	.1455E-09	.1269E-09
1.5	.1002E-02	.7997E-02	.1000E-02	.1000E-01	.1456E-09	.1269E-09
1.6	.1002E-02	.7997E-02	.1000E-02	.1000E-01	.1456E-09	.1269E-09
1.7	.1002E-02	.7997E-02	.1000E-02	.1000E-01	.1457E-09	.1270E-09
1.8	.1002E-02	.7997E-02	.1000E-02	.1000E-01	.1457E-09	.1270E-09
1.9	.1002E-02	.7997E-02	.1000E-02	.1000E-01	.1458E-09	.1272E-09

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals using Cowell's 3-points usual and the corresponding  $\beta$ -optimal method methods with restriction, in  $H_{a,b}^{(4)}$ -space, at the nodal points, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.833333$ ,  $\beta_1=.833333$ ,  $\beta_2=.833333$ .



Table - 6

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.9	.8363E-03	.8327E-02	.8363E-03	.1000E-01	.8830E-13	.8736E-13
-1.8	.8361E-03	.8328E-02	.8361E-03	.1000E-01	.8795E-13	.8708E-13
-1.7	.8360E-03	.8328E-02	.8360E-03	.1000E-01	.8759E-13	.8676E-13
-1.6	.8360E-03	.8328E-02	.8359E-03	.1000E-01	.8816E-13	.8735E-13
-1.5	.8360E-03	.8328E-02	.8359E-03	.1000E-01	.8860E-13	.8779E-13
-1.4	.8361E-03	.8328E-02	.8359E-03	.1000E-01	.8861E-13	.8779E-13
-1.3	.8361E-03	.8328E-02	.8360E-03	.1000E-01	.8776E-13	.8693E-13
-1.2	.8362E-03	.8328E-02	.8361E-03	.1000E-01	.8804E-13	.8718E-13
-1.1	.8363E-03	.8328E-02	.8362E-03	.1000E-01	.8808E-13	.8720E-13
-1.0	.8364E-03	.8327E-02	.8362E-03	.1000E-01	.8705E-13	.8613E-13
-.9	.8365E-03	.8327E-02	.8363E-03	.1000E-01	.8773E-13	.8679E-13
-.8	.8366E-03	.8327E-02	.8364E-03	.1000E-01	.9176E-13	.9078E-13
-.7	.8367E-03	.8327E-02	.8365E-03	.1000E-01	.9188E-13	.9089E-13
-.6	.8368E-03	.8327E-02	.8366E-03	.1000E-01	.8998E-13	.8896E-13
-.5	.8368E-03	.8327E-02	.8366E-03	.1000E-01	.9181E-13	.9078E-13
-.4	.8369E-03	.8326E-02	.8367E-03	.1000E-01	.8925E-13	.8820E-13
-.3	.8369E-03	.8326E-02	.8367E-03	.1000E-01	.8664E-13	.8557E-13
-.2	.8369E-03	.8326E-02	.8367E-03	.1000E-01	.9066E-13	.8960E-13
-.1	.8369E-03	.8326E-02	.8367E-03	.1000E-01	.9529E-13	.9424E-13
.0	.8369E-03	.8326E-02	.8367E-03	.1000E-01	.9013E-13	.8910E-13
.1	.8368E-03	.8326E-02	.8367E-03	.1000E-01	.8803E-13	.8697E-13
.2	.8368E-03	.8327E-02	.8367E-03	.1000E-01	.9601E-13	.9496E-13
.3	.8368E-03	.8327E-02	.8366E-03	.1000E-01	.7816E-13	.7714E-13
.4	.8367E-03	.8327E-02	.8366E-03	.1000E-01	.8363E-13	.8258E-13
.5	.8367E-03	.8327E-02	.8365E-03	.1000E-01	.9096E-13	.8993E-13
.6	.8366E-03	.8327E-02	.8365E-03	.1000E-01	.1108E-12	.1099E-12
.7	.8366E-03	.8327E-02	.8365E-03	.1000E-01	.1309E-12	.1298E-12
.8	.8365E-03	.8327E-02	.8364E-03	.1000E-01	.6498E-13	.6401E-13
.9	.8365E-03	.8327E-02	.8364E-03	.1000E-01	.7668E-13	.7575E-13
1.0	.8364E-03	.8327E-02	.8363E-03	.1000E-01	.7596E-13	.7524E-13
1.1	.8364E-03	.8327E-02	.8363E-03	.1000E-01	.6712E-13	.6632E-13
1.2	.8363E-03	.8327E-02	.8362E-03	.1000E-01	.1469E-12	.1463E-12
1.3	.8363E-03	.8328E-02	.8362E-03	.1000E-01	.1283E-12	.1272E-12
1.4	.8362E-03	.8328E-02	.8361E-03	.1000E-01	.9255E-13	.9159E-13
1.5	.8362E-03	.8328E-02	.8361E-03	.1000E-01	.1193E-12	.1183E-12
1.6	.8361E-03	.8328E-02	.8360E-03	.1000E-01	.2372E-12	.2358E-12
1.7	.8361E-03	.8328E-02	.8360E-03	.1000E-01	.7615E-13	.7538E-13
1.8	.8360E-03	.8328E-02	.8359E-03	.1000E-01	.6879E-13	.6732E-13
1.9	.8360E-03	.8328E-02	.8359E-03	.1000E-01	.3280E-12	.3273E-12

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals using Cowell's 3-points  $u_{\text{usual}}$  and the corresponding  $\beta$ -optimal methods with restriction, in  $H_{a,b}^{(5)}$ -space, at the nodal points, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.833333$ ,  $\beta_1=.833333$ ,  $\beta_2=.833333$ .

Table - 5(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.368457E-02
2	.961538E-02	.753787E-02
3	.769231E-02	.604319E-02
4	.448718E-02	.354015E-02
5	.108974E-01	.808629E-02
6	.466667E-02	.355873E-02
7	.102985E-01	.981491E-02
8	.749346E-02	.687194E-02
9	.539873E-02	.461103E-02
10	.270475E-02	.147793E-02

Table for average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal method with restriction, in  $H_{a,b}^{(4)}$ -space, on 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

Table - 6(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.478223E-02
2	.961538E-02	.957449E-02
3	.769231E-02	.766364E-02
4	.448718E-02	.447275E-02
5	.108974E-01	.108284E-01
6	.466667E-02	.464119E-02
7	.102985E-01	.102867E-01
8	.749346E-02	.747828E-02
9	.539873E-02	.537951E-02
10	.270475E-02	.267498E-02

Table for average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods with restriction, in  $H_{a,b}^{(5)}$ -space, on 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

#### 6.4 Optimal Multistep Methods in $H_{a,b}^{(m)}$ -Space Interpolatory for Polynomials.

In section 4.4, we have given an approach for determining the optimal multistep methods interpolatory for polynomials of certain degree, in  $H^2(C_r)$  space. In  $H_{a,b}^{(m)}$  space, the equispaced nodal points  $x_{n+k-1}$ ,  $i=0(1)k$  are along a straight line in an interval  $[a,b]$ . To find the  $\beta$ -optimal methods interpolatory for polynomials of certain degree we have to solve the system of equations,  $\hat{C} \hat{b} = \hat{d}$ ,

where  $\hat{b} = \left( \hat{\gamma}_{q-1,n}, \hat{\gamma}_{q,n}, \dots, \hat{\gamma}_{k-\delta_{t_0},n} \right)^T$ ,

$$\hat{c}_{ij} = h^2 \sum_{m=0}^j \sum_{l=0}^i (-1)^{l+m} C_l^j C_m^i D_2''(x_{n+k-\delta_{t_0}-m}, \bar{x}_{n+k-\delta_{t_0}-l}),$$

$$\hat{d}_i = -h^2 \sum_{j=0}^{q-2} \gamma_j^u \sum_{m=0}^j \sum_{l=0}^i (-1)^{l+m} C_l^j C_m^i D_2''(x_{n+k-\delta_{t_0}-m}, \bar{x}_{n+k-\delta_{t_0}-l})$$

$$- \sum_{j=0}^k \alpha_j \sum_{l=0}^i (-1)^{l+i} C_l^i D_2(x_{n+k-j}, \bar{x}_{n+k-\delta_{t_0}-l}), \quad i, j = q-1(1)k-\delta_{t_0},$$

where  $\gamma_j^u$ ,  $j=0(1)q-2$  are equal to  $\beta_j$ ,  $j=0(1)q-2$  in the usual method and  $D_2(y, x)$  and  $D_2''(y, x)$  are given by (18) and (19).

In  $H_{a,b}^{(3)}$ -space, the numerical values of optimal beta coefficients and the square norm of local truncation error functionals for usual and  $\beta$ -optimal methods interpolatory for polynomials of degree 3 are independent of points and numerically these quantities are nearly the same as in the case of  $\beta$ -optimal method in  $H_{a,b}^{(3)}$ -space, presented in section 6.3. The corresponding table, therefore is omitted.

Table - 7(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.244860E-13
2	.961538E-02	.470848E-13
3	.769231E-02	.169158E-12
4	.448718E-02	.274652E-13
5	.108974E-01	.115902E-12
6	.466667E-02	.101750E-12
7	.236952E-01	.222815E-01
8	.102985E-01	.841119E-02
9	.749346E-02	.507054E-02
10	.539873E-02	.232991E-02
11	.270475E-02	.207524E-02
12	.187310E-02	.402683E-02
13	.127316E-02	.596774E-02

Table for average discretisation errors using Cowell's 3-points usual and  $\beta$ -optimal method interpolatory for polynomials of degree 3, in  $H_{a,b}^{(3)}$ -space, on 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

Table - 8

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.9	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.8	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.7	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.6	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.5	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.4	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.3	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.2	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.1	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-1.0	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.9	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.8	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.7	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.6	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.5	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.4	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.3	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.2	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
-.1	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.0	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.1	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.2	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.3	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1269E-09
.4	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.5	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.6	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.7	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.8	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
.9	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
1.0	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
1.1	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1454E-09	.1269E-09
1.2	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1454E-09	.1269E-09
1.3	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1269E-09
1.4	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1455E-09	.1270E-09
1.5	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1456E-09	.1270E-09
1.6	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1456E-09	.1271E-09
1.7	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1457E-09	.1271E-09
1.8	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1457E-09	.1272E-09
1.9	.1000E-02	.8000E-02	.1000E-02	.1000E-01	.1458E-09	.1273E-09

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods interpolatory for polynomials of degree 3, in  $H_{a,b}^{(4)}$ -space, at the nodal points, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,

$h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.833333$ ,  $\beta_1=.833333$ ,  $\beta_2=.833333$ ,  $\gamma_1=1$ ,  $\gamma_2=-1$ ,  $\gamma_3=1/12$ .

Table - 9

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.9	.8363E-03	.8327E-02	.8363E-03	.1000E-01	.8830E-13	.8736E-13
-1.8	.8360E-03	.8328E-02	.8360E-03	.1000E-01	.8795E-13	.8710E-13
-1.7	.8358E-03	.8328E-02	.8358E-03	.1000E-01	.8759E-13	.8681E-13
-1.6	.8356E-03	.8329E-02	.8356E-03	.1000E-01	.8816E-13	.8743E-13
-1.5	.8355E-03	.8329E-02	.8355E-03	.1000E-01	.8860E-13	.8792E-13
-1.4	.8353E-03	.8329E-02	.8353E-03	.1000E-01	.8861E-13	.8797E-13
-1.3	.8352E-03	.8330E-02	.8352E-03	.1000E-01	.8776E-13	.8716E-13
-1.2	.8351E-03	.8330E-02	.8351E-03	.1000E-01	.8804E-13	.8747E-13
-1.1	.8350E-03	.8330E-02	.8350E-03	.1000E-01	.8808E-13	.8754E-13
-1.0	.8349E-03	.8330E-02	.8349E-03	.1000E-01	.8705E-13	.8654E-13
-.9	.8349E-03	.8330E-02	.8349E-03	.1000E-01	.8773E-13	.8725E-13
-.8	.8348E-03	.8330E-02	.8348E-03	.1000E-01	.9176E-13	.9129E-13
-.7	.8347E-03	.8331E-02	.8347E-03	.1000E-01	.9188E-13	.9144E-13
-.6	.8347E-03	.8331E-02	.8347E-03	.1000E-01	.8998E-13	.8956E-13
-.5	.8346E-03	.8331E-02	.8346E-03	.1000E-01	.9181E-13	.9140E-13
-.4	.8346E-03	.8331E-02	.8346E-03	.1000E-01	.8925E-13	.8884E-13
-.3	.8345E-03	.8331E-02	.8345E-03	.1000E-01	.8664E-13	.8627E-13
-.2	.8345E-03	.8331E-02	.8345E-03	.1000E-01	.9066E-13	.9031E-13
-.1	.8344E-03	.8331E-02	.8344E-03	.1000E-01	.9529E-13	.9494E-13
.0	.8344E-03	.8331E-02	.8344E-03	.1000E-01	.9013E-13	.8980E-13
.1	.8344E-03	.8331E-02	.8344E-03	.1000E-01	.8803E-13	.8767E-13
.2	.8343E-03	.8331E-02	.8343E-03	.1000E-01	.9601E-13	.9568E-13
.3	.8343E-03	.8331E-02	.8343E-03	.1000E-01	.7816E-13	.7789E-13
.4	.8343E-03	.8331E-02	.8343E-03	.1000E-01	.8363E-13	.8331E-13
.5	.8342E-03	.8332E-02	.8342E-03	.1000E-01	.9096E-13	.9064E-13
.6	.8342E-03	.8332E-02	.8342E-03	.1000E-01	.1108E-12	.1106E-12
.7	.8342E-03	.8332E-02	.8342E-03	.1000E-01	.1309E-12	.1305E-12
.8	.8342E-03	.8332E-02	.8342E-03	.1000E-01	.6498E-13	.6464E-13
.9	.8342E-03	.8332E-02	.8342E-03	.1000E-01	.7668E-13	.7641E-13
1.0	.8341E-03	.8332E-02	.8341E-03	.1000E-01	.7596E-13	.7577E-13
1.1	.8341E-03	.8332E-02	.8341E-03	.1000E-01	.6712E-13	.6714E-13
1.2	.8341E-03	.8332E-02	.8341E-03	.1000E-01	.1469E-12	.1468E-12
1.3	.8341E-03	.8332E-02	.8341E-03	.1000E-01	.1283E-12	.1281E-12
1.4	.8341E-03	.8332E-02	.8341E-03	.1000E-01	.9255E-13	.9247E-13
1.5	.8340E-03	.8332E-02	.8340E-03	.1000E-01	.1193E-12	.1190E-12
1.6	.8340E-03	.8332E-02	.8340E-03	.1000E-01	.2372E-12	.2365E-12
1.7	.8340E-03	.8332E-02	.8340E-03	.1000E-01	.7615E-13	.7555E-13
1.8	.8340E-03	.8332E-02	.8340E-03	.1000E-01	.6879E-13	.6822E-13
1.9	.8340E-03	.8332E-02	.8340E-03	.1000E-01	.3280E-12	.3276E-12

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functionals using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for polynomials of degree 3, in  $H_{a,b}^{(5)}$ -space, at the nodal points, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.833333$ ,  $\beta_1=.833333$ ,  $\beta_2=.833333$ ,  $\gamma_0=1$ ,  $\gamma_1=-1$ ,  $\gamma_2=1/12$ .

Table - 8(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.384615E-02
2	.961538E-02	.769231E-02
3	.769231E-02	.615385E-02
4	.448718E-02	.358974E-02
5	.108974E-01	.871795E-02
6	.466667E-02	.373333E-02
7	.102985E-01	.992101E-02
8	.749346E-02	.700888E-02
9	.539873E-02	.478497E-02
10	.270475E-02	.174875E-02

Table for average discretisation errors using Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods interpolatory for polynomials of degree 3, in  $H_{a,b}^{(4)}$ -space, on 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

Table - 9(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.480769E-02	.480242E-02
2	.961538E-02	.960272E-02
3	.769231E-02	.768121E-02
4	.448718E-02	.447948E-02
5	.108974E-01	.108866E-01
6	.466667E-02	.466170E-02
7	.102985E-01	.102970E-01
8	.749346E-02	.749139E-02
9	.539873E-02	.539595E-02
10	.270475E-02	.269993E-02

Table for average discretisation errors using Cowell's 3-points usual method and the corresponding  $\beta$ -optimal method interpolatory for polynomials of degree 3, in  $H_{a,b}^{(5)}$ -space, on 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .

### 6.5 Optimal Multistep Methods in $H_{a,b}^{(m)}$ -Space Interpolatory for a Set of Preassigned Functions.

To obtain the coefficients  $\hat{\beta}_{j,n}^F$ ,  $j = \delta_{t_0}(1)k$  of a  $\beta$ -optimal multistep method

$$(23) \quad \sum_{j=0}^k \alpha_j y_{n+k-j} + h^2 \sum_{j=\delta_{t_0}}^k \hat{\beta}_{j,n}^F f_{n+k-j} = 0$$

corresponding to the usual method (21) of highest degree polynomial precision, in  $H_{a,b}^{(m)}$  space with prefixed  $\alpha_j$ 's, interpolatory for  $q$  number of linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$  we have to solve a system of linear equations given in the matrix form

$$\begin{bmatrix} A & F^* \\ F & 0 \end{bmatrix} \begin{bmatrix} b \\ \lambda \end{bmatrix} = \begin{bmatrix} c \\ f \end{bmatrix},$$

where  $A = (A_{ij})$ , with  $A_{ij} = D2''(x_{n+k-j}, x_{n+k-i})$ ,  $i, j = \delta_{t_0}(1)k$ ,

$F = (F_{ij})$ , with  $F_{ij} = f_i''(x_{n+k-j})$ ,  $i=1(1)q$ ,  $j=\delta_{t_0}(1)k$ ,

$$b = h^2 \left( \hat{\beta}_{\delta_{t_0}, n}^F, \hat{\beta}_{\delta_{t_0}+1, n}^F, \dots, \hat{\beta}_{k, n}^F \right)^T,$$

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_q)^T,$$

$$c = (c_{\delta_{t_0}}, \dots, c_k)^T; \text{ with } c_i = - \sum_{j=0}^k \alpha_j D2(x_{n+k-j}, x_{n+k-i}), \quad i=\delta_{t_0}(1)k,$$

and

$$f = (\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_q)^T; \quad \text{with } \tilde{f}_i = - \sum_{j=0}^k \alpha_j f_i(x_{n+k-j}).$$

This method is characterized as follows.

**Theorem 9:** The  $\beta$ -optimal multistep method (16), in  $H_{a,b}^{(m)}$  space, interpolatory for linearly independent arbitrary functions  $f_1, f_2, \dots, f_q$  is characterized by that it is locally interpolatory for the functions

$$\{ f_1, f_2, \dots, f_q \} \cup \{ h_i, i = \delta_{t_0}(1)k-q \},$$

$$\text{where } h_i(x) = D2(x, x_{n+k-1}) - \sum_{j=k-q+1}^k \bar{g}_{j+q-k, i+1-\delta_{t_0}} D2(x, x_{n+k-j}),$$

and

$$G = P^{-1}E = (g_{ij})_{\substack{i=1(1)q \\ j=1(1)k-q-\delta_{t_0}+1}},$$

$$P = \left( f''_1(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=k-q+1(1)k}}, \quad E = \left( f''_1(x_{n+k-j}) \right)_{\substack{i=1(1)q \\ j=\delta_{t_0}(1)k-q}},$$

where  $D2(x, x_{n+k-1})$ ,  $i = \delta_{t_0}(1)k-q$  are given by (18).

Proof: The proof follows from Theorem 8 of Chapter 1.

The following tables are for average discretisation errors for the usual method with highest degree polynomial precision and the beta optimal method interpolatory for linearly independent functions,  $\exp(3.6x)$  and  $\exp(-3.6x)$  on 6 BVP-s with differential equations of the form  $\frac{d^2y}{dx^2} = f(x, y) = g(x)$ , having the true solutions given as follows.

1.  $\exp(3.6x)$
2.  $\exp(-3.6x)$
3.  $\exp(3.59x)$
4.  $\exp(-3.59x)$
5.  $\exp(3.61x)$
6.  $\exp(-3.61x)$

For this implementation, we are presenting the numerical results for some more values of  $h$ .

Table-10(i)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.411111E-01	.298928E-11
2	.411111E-01	.563743E-13
3	.424271E-01	.313737E-01
4	.424271E-01	.261371E-01
5	.398346E-01	.300258E-01
6	.398346E-01	.250216E-01

Table-10(ii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.661409E+00	.118234E-11
2	.661409E+00	.443688E-13
3	.682516E+00	.797048E-01
4	.682516E+00	.455930E-01
5	.640932E+00	.762205E-01
6	.640932E+00	.436243E-01

In  $H_{a,b}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.1$

In  $H_{a,b}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.2$



Table-10(iii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.103140E+02	.398868E-13
2	.103140E+02	.812012E-13
3	.106393E+02	.168280E+00
4	.106393E+02	.102169E+00
5	.999829E+01	.160507E+00
6	.999829E+01	.973143E-01

In  $H_{a,b}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.4$

Table-11(i)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.411111E-01	.296837E-11
2	.411111E-01	.541711E-13
3	.424271E-01	.463545E-03
4	.424271E-01	.245588E-03
5	.398346E-01	.440861E-03
6	.398346E-01	.232666E-03

In  $H_{a,b}^{(4)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.1$

Table-11(iii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.103140E+02	.225625E-13
2	.103140E+02	.858510E-13
3	.106393E+02	.560321E-01
4	.106393E+02	.544066E-01
5	.999829E+01	.531115E-01
6	.999829E+01	.515581E-01

In  $H_{a,b}^{(4)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.4$

Table-12(i)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.411111E-01	.300638E-11
2	.411111E-01	.540470E-13
3	.424271E-01	.602654E-03
4	.424271E-01	.230377E-03
5	.398346E-01	.220966E-03
6	.398346E-01	.218073E-03

In  $H_{2,c}^{(5)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.1$

Table-10(iv)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.244472E+02	.825927E-14
2	.244472E+02	.881116E-13
3	.252119E+02	.250026E+00
4	.252119E+02	.175963E+00
5	.237048E+02	.238187E+00
6	.237048E+02	.167366E+00

In  $H_{a,b}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.5$

Table-11(ii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.661409E+00	.133426E-11
2	.661409E+00	.313977E-13
3	.682516E+00	.427492E-02
4	.682516E+00	.372993E-02
5	.640932E+00	.405298E-02
6	.640932E+00	.353237E-02

In  $H_{a,b}^{(4)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.2$

Table-11(iv)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.244472E+02	.799559E-14
2	.244472E+02	.245784E-13
3	.252119E+02	.125536E+00
4	.252119E+02	.123319E+00
5	.237048E+02	.119040E+00
6	.237048E+02	.116921E+00

In  $H_{a,b}^{(4)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.5$

Table-12(ii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.661409E+00	.118219E-11
2	.661409E+00	.478986E-13
3	.682516E+00	.615306E-03
4	.682516E+00	.364846E-02
5	.640932E+00	.349374E-02
6	.640932E+00	.345426E-02

In  $H_{2,c}^{(5)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.2$

Table-12(iii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.103140E+02	.290111E-13
2	.103140E+02	.123470E-12
3	.106393E+02	.636403E-03
4	.106393E+02	.534976E-01
5	.999829E+01	.511338E-01
6	.999829E+01	.506869E-01

In  $H_{2,c}^{(5)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.4$

Table-12(iv)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.244472E+02	.624203E-13
2	.244472E+02	.478742E-13
3	.252119E+02	.644939E-03
4	.252119E+02	.121250E+00
5	.237048E+02	.115745E+00
6	.237048E+02	.114938E+00

In  $H_{2,c}^{(5)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.5$

Table-13(i)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.411111E-01	.299115E-11
2	.411111E-01	.596382E-13
3	.424271E-01	.232631E-03
4	.424271E-01	.226527E-03
5	.398346E-01	.220219E-03
6	.398346E-01	.214386E-03

In  $H_{a,b}^{(6)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.1$

Table-13(ii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.661409E+00	.130226E-11
2	.661409E+00	.422933E-13
3	.682516E+00	.368304E-02
4	.682516E+00	.358649E-02
5	.640932E+00	.348718E-02
6	.640932E+00	.339492E-02

In  $H_{a,b}^{(6)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.2$

Table-13(iii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.103140E+02	.288784E-13
2	.103140E+02	.213345E-13
3	.106393E+02	.539174E-01
4	.106393E+02	.525345E-01
5	.999829E+01	.510863E-01
6	.999829E+01	.497645E-01

In  $H_{a,b}^{(6)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.4$

Table-13(iv)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.244472E+02	.252645E-12
2	.244472E+02	.183812E-12
3	.252119E+02	.122021E+00
4	.252119E+02	.118993E+00
5	.237048E+02	.115671E+00
6	.237048E+02	.112775E+00

In  $H_{a,b}^{(6)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.5$

However, the square norm of local truncation error functionals for  $\beta$ -optimal method interpolatory for  $\exp(\pm 3.6x)$  in  $H_{a,b}^{(4)}$ ,  $H_{a,b}^{(5)}$  and  $H_{a,b}^{(6)}$  spaces are not being less than those for the corresponding usual method with highest degree polynomial precision. Truly

speaking, these two kinds of usual and optimal methods are not comparable. These two methods will be comparable if we choose usual method as a method with function evaluation at three points which is interpolatory for  $x^2$  and the two other functions for which the optimal method is made interpolatory.

In the following tables we are presenting the numerical results for the usual method with function evaluation at three points which is interpolatory for  $x^2$ ,  $\cos(12x)$ ,  $\sin(12x)$  and the corresponding  $\beta$ -optimal method interpolatory for  $\cos(12x)$ ,  $\sin(12x)$  on 6 BVP-s with differential equations of the form  $\frac{d^2y}{dx^2} = f(x,y)$ , having the true solutions given as follows.

1.  $\sin 12x$
2.  $\cos 12x$
3.  $\sin 11.9x$
4.  $\cos 11.9x$
5.  $\sin 11.8x$
6.  $\cos 11.8x$

Although in  $H_{a,b}^{(3)}$ -space, the square norm of local truncation error functional for the usual method is close to that for the optimal method, the absolute average discretisation error for the optimal method is less than that for the usual method. But in  $H_{a,b}^{(m)}$ -space,  $m \geq 4$ , the square norm of local truncation error functionals for usual and optimal methods are numerically nearly the same, and the absolute average discretisation error for these two methods are almost same including some round-off errors. We therefore omit the corresponding tables.

Table - 14

$x_n$	$\beta_{0n}$	$\beta_{1n}$	$\beta_{2n}$	r-sum
-1.9	.896943E-01	.820611E+00	.896943E-01	.100410E-01
-1.8	.896943E-01	.820611E+00	.896943E-01	.100375E-01
-1.7	.896943E-01	.820611E+00	.896943E-01	.100345E-01
-1.6	.896943E-01	.820611E+00	.896943E-01	.100320E-01
-1.5	.896943E-01	.820611E+00	.896943E-01	.100298E-01
-1.4	.896943E-01	.820611E+00	.896943E-01	.100279E-01
-1.3	.896943E-01	.820611E+00	.896943E-01	.100263E-01
-1.2	.896943E-01	.820611E+00	.896943E-01	.100248E-01
-1.1	.896943E-01	.820611E+00	.896943E-01	.100234E-01
-1.0	.896943E-01	.820611E+00	.896943E-01	.100222E-01
-.9	.896943E-01	.820611E+00	.896943E-01	.100212E-01
-.8	.896943E-01	.820611E+00	.896943E-01	.100202E-01
-.7	.896943E-01	.820611E+00	.896943E-01	.100193E-01
-.6	.896943E-01	.820611E+00	.896943E-01	.100185E-01
-.5	.896943E-01	.820611E+00	.896943E-01	.100177E-01
-.4	.896943E-01	.820611E+00	.896943E-01	.100170E-01
-.3	.896943E-01	.820611E+00	.896943E-01	.100164E-01
-.2	.896943E-01	.820611E+00	.896943E-01	.100158E-01
-.1	.896943E-01	.820611E+00	.896943E-01	.100153E-01
.0	.896943E-01	.820611E+00	.896943E-01	.100147E-01
.1	.896943E-01	.820611E+00	.896943E-01	.100143E-01
.2	.896943E-01	.820611E+00	.896943E-01	.100138E-01
.3	.896943E-01	.820611E+00	.896943E-01	.100134E-01
.4	.896943E-01	.820611E+00	.896943E-01	.100130E-01
.5	.896943E-01	.820611E+00	.896943E-01	.100126E-01
.6	.896943E-01	.820611E+00	.896943E-01	.100123E-01
.7	.896943E-01	.820611E+00	.896943E-01	.100119E-01
.8	.896943E-01	.820611E+00	.896943E-01	.100116E-01
.9	.896943E-01	.820611E+00	.896943E-01	.100113E-01
1.0	.896943E-01	.820611E+00	.896943E-01	.100110E-01
1.1	.896943E-01	.820611E+00	.896943E-01	.100108E-01
1.2	.896943E-01	.820611E+00	.896943E-01	.100105E-01
1.3	.896943E-01	.820611E+00	.896943E-01	.100103E-01
1.4	.896943E-01	.820611E+00	.896943E-01	.100100E-01
1.5	.896943E-01	.820611E+00	.896943E-01	.100098E-01
1.6	.896943E-01	.820611E+00	.896943E-01	.100096E-01
1.7	.896943E-01	.820611E+00	.896943E-01	.100094E-01
1.8	.896943E-01	.820611E+00	.896943E-01	.100092E-01
1.9	.896943E-01	.820611E+00	.896943E-01	.100090E-01

Table for  $\beta$ -coefficients for the usual method with function evaluation at three points, interpolatory for  $x^2$ ,  $\cos(12x)$ ,  $\sin(12x)$  and their row-sum at the nodal points with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ .

Table - 15

$x_n$	$h^2 \hat{\beta}_{0n}$	$h^2 \hat{\beta}_{1n}$	$h^2 \hat{\beta}_{2n}$	r-sum	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.9	.9291E-03	.8183E-02	.9291E-03	.1004E-01	.5629E-06	.5611E-06
-1.8	.9264E-03	.8185E-02	.9264E-03	.1004E-01	.5629E-06	.5613E-06
-1.7	.9240E-03	.8186E-02	.9240E-03	.1003E-01	.5629E-06	.5614E-06
-1.6	.9220E-03	.8188E-02	.9220E-03	.1003E-01	.5629E-06	.5615E-06
-1.5	.9203E-03	.8189E-02	.9203E-03	.1003E-01	.5629E-06	.5616E-06
-1.4	.9188E-03	.8190E-02	.9188E-03	.1003E-01	.5629E-06	.5617E-06
-1.3	.9175E-03	.8191E-02	.9175E-03	.1003E-01	.5629E-06	.5618E-06
-1.2	.9164E-03	.8192E-02	.9164E-03	.1002E-01	.5629E-06	.5619E-06
-1.1	.9153E-03	.8193E-02	.9153E-03	.1002E-01	.5629E-06	.5619E-06
-1.0	.9144E-03	.8193E-02	.9144E-03	.1002E-01	.5629E-06	.5620E-06
-.9	.9135E-03	.8194E-02	.9135E-03	.1002E-01	.5629E-06	.5620E-06
-.8	.9128E-03	.8195E-02	.9128E-03	.1002E-01	.5629E-06	.5621E-06
-.7	.9121E-03	.8195E-02	.9121E-03	.1002E-01	.5629E-06	.5621E-06
-.6	.9114E-03	.8196E-02	.9114E-03	.1002E-01	.5629E-06	.5621E-06
-.5	.9109E-03	.8196E-02	.9109E-03	.1002E-01	.5629E-06	.5622E-06
-.4	.9103E-03	.8196E-02	.9103E-03	.1002E-01	.5629E-06	.5622E-06
-.3	.9098E-03	.8197E-02	.9098E-03	.1002E-01	.5629E-06	.5622E-06
-.2	.9093E-03	.8197E-02	.9093E-03	.1002E-01	.5629E-06	.5622E-06
-.1	.9089E-03	.8197E-02	.9089E-03	.1002E-01	.5629E-06	.5623E-06
.0	.9085E-03	.8198E-02	.9085E-03	.1001E-01	.5629E-06	.5623E-06
.1	.9081E-03	.8198E-02	.9081E-03	.1001E-01	.5629E-06	.5623E-06
.2	.9078E-03	.8198E-02	.9078E-03	.1001E-01	.5629E-06	.5623E-06
.3	.9074E-03	.8199E-02	.9074E-03	.1001E-01	.5629E-06	.5624E-06
.4	.9071E-03	.8199E-02	.9071E-03	.1001E-01	.5629E-06	.5624E-06
.5	.9068E-03	.8199E-02	.9068E-03	.1001E-01	.5629E-06	.5624E-06
.6	.9066E-03	.8199E-02	.9066E-03	.1001E-01	.5629E-06	.5624E-06
.7	.9063E-03	.8199E-02	.9063E-03	.1001E-01	.5629E-06	.5624E-06
.8	.9061E-03	.8200E-02	.9061E-03	.1001E-01	.5629E-06	.5624E-06
.9	.9058E-03	.8200E-02	.9058E-03	.1001E-01	.5629E-06	.5624E-06
1.0	.9056E-03	.8200E-02	.9056E-03	.1001E-01	.5629E-06	.5625E-06
1.1	.9054E-03	.8200E-02	.9054E-03	.1001E-01	.5629E-06	.5625E-06
1.2	.9052E-03	.8200E-02	.9052E-03	.1001E-01	.5629E-06	.5625E-06
1.3	.9050E-03	.8200E-02	.9050E-03	.1001E-01	.5629E-06	.5625E-06
1.4	.9048E-03	.8200E-02	.9048E-03	.1001E-01	.5629E-06	.5625E-06
1.5	.9046E-03	.8201E-02	.9046E-03	.1001E-01	.5629E-06	.5625E-06
1.6	.9045E-03	.8201E-02	.9045E-03	.1001E-01	.5629E-06	.5625E-06
1.7	.9043E-03	.8201E-02	.9043E-03	.1001E-01	.5629E-06	.5625E-06
1.8	.9041E-03	.8201E-02	.9041E-03	.1001E-01	.5629E-06	.5625E-06
1.9	.9040E-03	.8201E-02	.9040E-03	.1001E-01	.5629E-06	.5625E-06

Table for optimal  $\beta$ 's, their row-sum and the square norm of local truncation error functional for Cowell's 3-points usual and the corresponding  $\beta$ -optimal methods, interpolatory for  $\cos(12x)$ ,  $\sin(12x)$ , in  $H_{a,b}^{(3)}$  space, at the nodal points, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_0=.833333$ ,  $\beta_1=.833333$ ,  $\beta_2=.833333$ .

Table-15(i)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.414802E-14	.311327E-14
2	.285788E-13	.593124E-14
3	.116894E-03	.955821E-04
4	.107057E-03	.862352E-04
5	.234438E-03	.190407E-03
6	.201188E-03	.163068E-03

In  $H_{2,c}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.1$

Table-15(ii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.312807E-13	.183479E-14
2	.736472E-14	.597840E-14
3	.277344E-02	.279421E-02
4	.261586E-02	.263423E-02
5	.546854E-02	.551106E-02
6	.465446E-02	.468707E-02

In  $H_{2,c}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.2$

Table-15(iii)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.228048E-13	.770885E-14
2	.255243E-13	.957722E-14
3	.433940E+00	.431640E+00
4	.349018E+00	.347560E+00
5	.815712E+00	.811524E+00
6	.580495E+00	.578334E+00

In  $H_{2,c}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.4$

Table-15(iv)

eqn	$\bar{e}_u$	$\bar{e}_o$
1	.287833E-13	.385882E-14
2	.208881E-12	.249007E-14
3	.115117E+02	.244841E+00
4	.681617E+02	.422203E+00
5	.270802E+02	.574618E+00
6	.136431E+03	.840303E+00

In  $H_{2,c}^{(3)}$ -space,  $a=-2$ ,  $b=2$ ,  $h=.5$

Now in Table 16, we are presenting the numerical results for Stormer's usual method with function evaluation at one point and the corresponding  $\beta$ -optimal method on 2 BVP-s with differential equations of the form  $\frac{d^2 y}{dx^2} = f(x,y) = g(x)$ , whose exact solutions are given by:

$$(1) \quad y(x) = \sin 4x,$$

$$(2) \quad y(x) = \cos 4x.$$

In  $H_{a,b}^{(m)}$  space with  $m \geq 4$ , the numerical results suggest that there is no remarkable difference in the square norm of local truncation error functionals for the above usual method and the optimal method. So, in this case we are interested in the numerical results for  $H_{a,b}^{(3)}$  space only.

Table - 16

$x_n$	$h^2 \hat{\beta}_{1n}$	$\ T_n\ ^2$	$\ \hat{T}_n\ ^2$
-1.9	.9848E-02	.1000E-05	.9747E-06
-1.8	.9861E-02	.1000E-05	.9769E-06
-1.7	.9872E-02	.1000E-05	.9786E-06
-1.6	.9881E-02	.1000E-05	.9802E-06
-1.5	.9889E-02	.1000E-05	.9815E-06
-1.4	.9896E-02	.1000E-05	.9826E-06
-1.3	.9902E-02	.1000E-05	.9837E-06
-1.2	.9907E-02	.1000E-05	.9846E-06
-1.1	.9912E-02	.1000E-05	.9854E-06
-1.0	.9917E-02	.1000E-05	.9861E-06
-.9	.9921E-02	.1000E-05	.9868E-06
-.8	.9924E-02	.1000E-05	.9874E-06
-.7	.9928E-02	.1000E-05	.9879E-06
-.6	.9931E-02	.1000E-05	.9884E-06
-.5	.9933E-02	.1000E-05	.9889E-06
-.4	.9936E-02	.1000E-05	.9893E-06
-.3	.9938E-02	.1000E-05	.9897E-06
-.2	.9940E-02	.1000E-05	.9901E-06
-.1	.9943E-02	.1000E-05	.9904E-06
.0	.9944E-02	.1000E-05	.9907E-06
.1	.9946E-02	.1000E-05	.9910E-06
.2	.9948E-02	.1000E-05	.9913E-06
.3	.9949E-02	.1000E-05	.9916E-06
.4	.9951E-02	.1000E-05	.9918E-06
.5	.9952E-02	.1000E-05	.9921E-06
.6	.9954E-02	.1000E-05	.9923E-06
.7	.9955E-02	.1000E-05	.9925E-06
.8	.9956E-02	.1000E-05	.9927E-06
.9	.9957E-02	.1000E-05	.9929E-06
1.0	.9958E-02	.1000E-05	.9931E-06
1.1	.9959E-02	.1000E-05	.9932E-06
1.2	.9960E-02	.1000E-05	.9934E-06
1.3	.9961E-02	.1000E-05	.9935E-06
1.4	.9962E-02	.1000E-05	.9937E-06
1.5	.9963E-02	.1000E-05	.9938E-06
1.6	.9964E-02	.1000E-05	.9940E-06
1.7	.9965E-02	.1000E-05	.9941E-06
1.8	.9965E-02	.1000E-05	.9942E-06
1.9	.9966E-02	.1000E-05	.9943E-06

Table for optimal  $\beta$  and the square norm of local truncation error functional for Stormer's 1-point usual and the corresponding  $\beta$ -optimal methods in  $H_{a,b}^{(3)}$  space, at the nodal points, with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ ,  $\alpha_0=-1$ ,  $\alpha_1=2$ ,  $\alpha_2=-1$ ,  $\beta_1=1$ .

Table - 16(i)

eqn.	$\bar{e}_u$	$\bar{e}_o$
1	.985155E-02	.594575E-02
2	.910997E-02	.540173E-02

Table for average discretisation errors using Stormer's 1-point usual and the corresponding  $\beta$ -optimal methods in  $H_{a,b}^{(3)}$ -space, on 10 BVP-s with  $a=-2$ ,  $b=2$ ,  $n=40$ ,  $h=.1$ .



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